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Abstract

The transient heat transfer in a solid undergoing ablation is a nonlinear problem, which involves a moving boundary that is not known a priori. In this paper the ablation problem is solved with constant material properties and time-variable heat flux using the integral method. An approximate, analytical, closed solution is obtained. The results are compared with solutions presented by the literature at constant and variable heat flux.

Nomenclature

$d_P(t)$	Heat Penetration Depth
$d_A(t)$	Ablation Depth
$u(t)$	Relative Depth
t_A	Ablation Time
T_A	Ablation Temperature
T_0	Initial Temperature
k	Thermal Conductivity
r	Density
c_P	Specific Heat at Constant Pressure
l	Heat of Ablation
n	Function Degree
$q'(t)$	Heat Flux
n	Inverse Stefan Number
t	Auxiliary variable
x	Space Coordinate
t	Time Coordinate

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Introduction

Transient heat conduction in a solid undergoing ablation represents an area of great technological importance. Problems of this type are inherently nonlinear and involve a moving boundary that is not known a priori. According to Chung[1] and Zien[2], the exact analytical solution for transient heat transfer in a solid undertaking ablation is very difficult and practically nonexistent. Only numerical and approximate analytical solutions have been made available and they necessarily require considerable numerical computation, even if a simplified model of the problem is used in the study. This work makes use of the integral method[3] to get a closed form, approximate, analytical solution for the phase-change ablation problem with time-variable heat flux.

Literature Review

Landau[4] first proposed an idealized ablation problem and solved it for the case of a semi-infinite melting solid with constant properties and with its face heated at constant rate. He applied numerical integration for his solution.

Sunderland and Grosh[5] presented the same problem but described a method of solution using finite differences for the case where the surface is heated by convection. Biot and Agrawal[6] used the variational method for the analysis of ablation with variable properties. Blackwell[7] used the finite volume method with exponential interpolation functions to solve Landau's problem.

Storti[8] considered a one-phase ablation problem as a two-phase Stefan one by the introduction of a fictitious phase occupying the region where the material has been removed. He solved this problem by the finite element method.

Goodman[3] solved Landau's problem using the heat balance integral method with a quadratic temperature profile for a constant heat flux. Zien[2] solved Landau's problem for two specific forms of heat flux with a refined heat balance method using exponential temperature profiles.

Physical Model

When a satellite, returning from its orbit around the Earth, reaches the atmosphere, a complex set of physical-chemical phenomena takes place at its surroundings. The majority of these phenomena are extremely exothermic and a part of generated heat achieves the satellite's surface, increasing its temperature.

To protect the satellite payload from critical temperature rises, a thermal protection system is used. The ablation protection system is one of them, and it is based on the phase-change phenomenon that occurs on the ablative material surface. For the present analysis the following simplifications are considered, as suggested by Landau[4]:

- The heat transfer problem is considered one-dimensional.
- The ablative material properties do not present considerable thermophysical modifications during the heating process, until it reaches the ablation temperature.
- All physical-chemical phenomena are considered known and given by a time-variable heat flux;
- All material, liquid or gas, produced during ablation is immediately removed from the surface and do not influence the thermal behavior of the protection system or of the satellite.

Based on these simplifications the physical model adopted consists of a semi-infinite ablative material which is heated in its surface, by a spatially uniform and time-variable heat source. In the beginning, the heat penetrates the material, raising its temperature in a region close to its surface. The length of this region is called heat penetration depth, $d_P(t)$, where $d_P(0)=0$. The heating continues until the front face temperature, $T(d_A(t),t)$, reaches the ablating temperature level (T_A) and causes the start up of the surface ablation. During the ablation, part of the heat is used to keep the surface at the ablating temperature $(T(d_A(t),t)=T_A)$ and the remaining heat is used to change the phase of the ablation material. The phase-change phenomenon consumes part of the virgin material. The length of this part is denominated ablation depth, $d_A(t)$, where $d_A(t_A)=0$ and t_A is the ablation time, i.e., the time in which $T(d_A(t_A),t_A)$ reaches T_A. Figure 1 shows a schematic of this physical model.

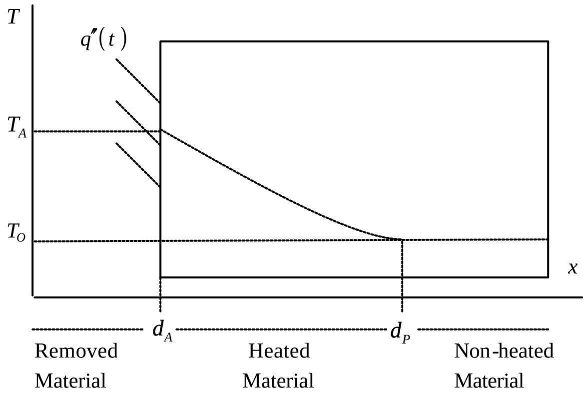

Fig.1 – Physical model adopted

Analytical Model

General Formulation

The following one-dimensional partial differential heat transfer equation is used to determine the ablation rate and the heat penetration depth:

$$r\, c_p \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T(x,t)}{\partial x}\right) \qquad (01)$$

This equation is integrated in x from $d_A(t)$ to $d_P(t)$. The results are rearranged using Leibniz's integral formula, resulting in:

$$r\, c_P \frac{d}{dt}\int_{d_A(t)}^{d_P(t)} T(x,t)\,dx - r\, c_P T(d_P(t),t)\frac{d d_P(t)}{dt} + r\, c_P T(d_A(t),t)\frac{d d_A(t)}{dt} = k\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=d_P(t)} - k\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=d_A(t)}. \qquad (02)$$

The following temperature profile is considered:

$$T(x,t) = A(t)\left(\frac{d_P(t)-x}{d_P(t)-d_A(t)}\right)^n + B(t), \qquad (03)$$

where *A(t)* and *B(t)* are time-dependent parameters used to adjust the adopted temperature profile to the problem solution.

Pre-Ablation Problem

The following conditions can be considered for the pre-ablation period:

$$\frac{d\,d_A(t)}{d\,t}=0\rightarrow d_A(t)=d_A, \qquad (04)$$

$$-k\frac{\partial T(x,t)}{\partial x}=q''(t) \qquad ,x=d_A(t), \qquad (05)$$

$$-k\frac{\partial T(x,t)}{\partial x}=0 \qquad ,\ x=d_P(t), \qquad (06)$$

$$T(x,t)=T_O \qquad ,\ x=d_P(t), \qquad (07)$$

$$T(x,t)=T_O \qquad ,\ t=t_O. \qquad (08)$$

Actually, Eq. 4 does not establishes a boundary condition, it just estates that no time variation of the ablation depth is equivalent to the situation where d_A is considered a constant value.

Solving Eq. 3, for $A(t)$ and $B(t)$ using the boundary conditions 5 and 6 indicated above, it is possible to obtain

$$T(x,t)=\frac{q''(t)(d_P(t)-d_A)}{k\,n}\left(\frac{d_P(t)-x}{d_P(t)-d_A}\right)^n+T_O. \qquad (09)$$

Substituting Eqs. 4, 5, 6, 7 and 9 in Eq. 2, solving the integral and rearranging it, one can obtain:

$$r\,c_P\frac{d}{d\,t}\left[\frac{q''(t)(d_P(t)-d_A)^2}{k\,n}+T_O\,(d_P(t)-d_A)\right] -r\,c_P T_O\frac{d\,d_P(t)}{d\,t}=q''(t). \qquad (10)$$

Simplifying Eq.(10),

$$r\,c_P\frac{d}{d\,t}\left(\frac{q''(t)(d_P(t)-d_A)^2}{k\,n(n+1)}\right)=q''(t). \qquad (11)$$

The solution of Eq. 11 with $t_O=0$ and $d_P(t_O)=d_A$ yields

$$d_P(t)=d_A+\sqrt{\left(\frac{k\,n(n+1)}{r\,c_P\,q''(t)}\right)\int_0^t q''(t)dt}\,. \qquad (12)$$

With the substitution of Eq. 12 in Eq. 9, and after some simplification, the following expression is obtained:

$$T(x,t)=\frac{q''(t)\left(d_A+\sqrt{\left(\frac{k\,n(n+1)}{r\,c_P\,q''(t)}\right)\int_0^t q''(t)dt}-x\right)^n}{k\,n\left(\sqrt{\left(\frac{k\,n(n+1)}{r\,c_P\,q''(t)}\right)\int_0^t q''(t)dt}\right)^{n-1}}+T_O\,. \qquad (13)$$

As $T(d_A(t_A),t_A)=T_A$, after some algebraic manipulation of Eq. 13, one gets

$$\frac{n}{(n+1)}r\,c_P\,k(T_A-T_O)^2=q''(t_A)\int_0^{t_A}q''(t)dt\,. \qquad (14)$$

In the case of constant heat flux, $q''(t)=q''$, Eq. 14 can be rewritten as

$$\frac{n}{(n+1)}r\,c_P\,k\frac{(T_A-T_O)^2}{q''^2}=t_A\,. \qquad (15)$$

According to Carslaw and Jaeger[9], the t_A expression obtained for an infinite solid heated in its surface by a constant heat flux is:

$$t_A=\frac{p}{4}k\,r\,c_P\left(\frac{(T_A-T_O)}{q''}\right)^2. \qquad (16)$$

Through the comparison of Eqs. 16 and 15, one can obtain

$$n=\frac{p}{(4-p)}=3,659792369\,. \qquad (17)$$

Therefore, for the pre-ablation period, $t\le t_A$, one has

$$T(x,t)=\begin{cases}\frac{q''(t)(d_P(t)-d_A)}{k\,n}\left(\frac{d_P(t)-x}{d_P(t)-d_A}\right)^n+T_O,\ x\le d_P(t)\\ T_O,\ \ x>d_P(t)\end{cases}. \qquad (18)$$

Ablation Problem

For the ablation period, the following boundary conditions can be considered:

$$T(x,t)=T_A \qquad ,x=d_A(t), \qquad (19)$$

$$-k\frac{\partial T(x,t)}{\partial x}=q''(t)-r\,l\frac{d\,d_A(t)}{d\,t} \qquad ,\ x=d_A(t), \qquad (20)$$

$$-k\frac{\partial T(x,t)}{\partial x}=0 \qquad ,\ x=d_P(t), \qquad (21)$$

$$T(x,t)=T_O \qquad , \; x=d_P(t)\cdot \tag{22}$$

The boundary given by Eq. 20 represents a heat balance between the net heat conducted through the virgin material, which is given by the difference between the heat source and the heat used to ablate the material. Solving Eq. 3 using Eqs. 19, 21 and 22, one can write

$$T(x,t)=(T_A-T_O)\left(\frac{d_P(t)-x}{d_P(t)-d_A(t)}\right)^n+T_O\,. \tag{23}$$

With the substitution of Eqs. 19, 21, 22 and 23 in Eq. 2 and in Eq. 20, it is obtained, respectively:

$$\begin{aligned} &r\,c_P\frac{d}{d\,t}\left[(T_A-T_O)(d_P(t)-d_A(t))+T_O(d_P(t)-d_A(t))\right] \\ &-r\,c_P T_O\frac{d\,d_P(t)}{d\,t}+r\,c_P T_A\frac{d\,d_A(t)}{d\,t}=\frac{k\,n(T_A-T_O)}{(d_P(t)-d_A(t))}\,, \end{aligned} \tag{24}$$

$$\frac{d\,d_A(t)}{d\,t}=\frac{q''(t)}{r\;l}-\frac{k\,n(T_A-T_O)}{r\;l\;(d_P(t)-d_A(t))}. \tag{25}$$

Collecting the similar terms and simplifying Eq. 24, one gets

$$\frac{d}{dt}(d_P(t)-d_A(t))+(n+1)\frac{d\,d_A(t)}{d\,t}=\frac{k\,n(n+1)}{r\,c_P\,(d_P(t)-d_A(t))}\,, \tag{26}$$

Equation 25 is, then, substituted in Eq. 26 and after the collection of similar terms, one can obtain

$$\begin{aligned} &\frac{d}{d\,t}(d_P(t)-d_A(t))= \\ &\frac{k\,n(n+1)(T_A-T_O)}{r\;l\;(d_P(t)-d_A(t))}\left(\frac{l}{c_P\,(T_A-T_O)}+1\right)-\frac{(n+1)q''(t)}{r\;l}. \end{aligned} \tag{27}$$

A new parameter t, defined as: $t\equiv\int_{t_A}^{t}\frac{(n+1)q''(t)}{r\;l}dt$ is introduced. Both penetration depths ($d_P(t)$ and $d_A(t)$) can be written as a function of this new variable. Hence, it is possible to rewrite the previous equation as

$$\begin{aligned} &\frac{d}{d\,t}(d_P(t)-d_A(t))\frac{d\,t}{d\,t}= \\ &\frac{k\,n(n+1)(T_A-T_O)}{r\;l\;(d_P(t)-d_A(t))}\left(\frac{l}{c_P\,(T_A-T_O)}+1\right)-\frac{d\,t}{d\,t}. \end{aligned} \tag{28}$$

Rearranging it,

$$\begin{aligned} &\frac{d}{d\,t}(d_P(t)-d_A(t))= \\ &\frac{k\,n(T_A-T_O)}{q''(t)(d_P(t)-d_A(t))}\left(\frac{l}{c_P\,(T_A-T_O)}+1\right)-1\cdot \end{aligned} \tag{29}$$

Defining the relative depth as $u(t)\equiv d_P(t)-d_A(t)$, the inverse Stefan number as $n\equiv\frac{l}{c_P\,(T_A-T_O)}$ and $F(t)\equiv\frac{k\,n\,(T_A-T_O)(n+1)}{q''(t)}$, one can finally write the equation as

$$\frac{d\,u(t)}{d\,t}=\frac{F(t)}{u(t)}-1\,\cdot \tag{30}$$

Physically, the inverse Stefan number indicates how much energy is consumed at the ablation phase-change process in comparison with the thermal storage capacity of the material. The relative depth represents the length of the heated material, as shown at Fig. 1. Actually, a temperature increase is observed in this region of the material due to the heat flux boundary.

Solving Eq. (30) for $u(t)$, from 0 to t, and using the initial condition $u(0)=u_A=d_P(t_A)-d_A(t_A)=\frac{k\,n(T_A-T_O)}{q''(t_A)}$, calculated from the algebraic manipulation of Eq. 9, after the application the boundary condition given by Eq. 19, for $t=t_A$, one gets:

$$u(t)=F(t)\left(LambertW\left\{\left(\frac{u_A}{F(t)}-1\right)\exp\left(\frac{u_A}{F(t)}-1-\frac{t}{F(t)}\right)\right\}+1\right)\cdot \tag{31}$$

Substituting the expressions for u_A, t and $F(t)$, and after some algebraic manipulation, the following equation is obtained:

$$\begin{aligned} u(t)=\;&\frac{k\,n\,(T_A-T_O)(n+1)}{q''(t)}\cdot \\ &\left(LambertW\left\{\begin{aligned}&\left(\frac{q''(t)-q''(t_A)}{q''(t_A)(n+1)}-\frac{n}{(n+1)}\right)\cdot \\ &\exp\left(\begin{aligned}&\frac{q''(t)-q''(t_A)}{q''(t_A)(n+1)}-\frac{n}{(n+1)} \\ &-\frac{q''(t)t}{k\,n\,(T_A-T_O)(n+1)}\end{aligned}\right)\end{aligned}\right\}+1\right)\cdot \end{aligned} \tag{32}$$

Now, getting back to Eq. 25 and introducing the t and $u(t)$ expressions, the following equation is resulted:

$$\frac{d\,d_A(t)}{d\,t} = \frac{1}{(n+1)} - \frac{k\,n(T_A - T_O)}{(n+1)\,q''(t)\,u(t)}. \qquad (33)$$

Solving Eq. 33, using Eq. 31 and using the initial condition : $d_A(0) = d_A$, one gets:

$$d_A(t) = d_A + \frac{t}{(n+1)} + \frac{k\,n(T_A - T_O)}{(n+1)\,q''\,F(t)} \cdot \left(u_A - t - F(t)\left(LambertW\left\{ \left(\frac{u_A}{F(t)} - 1 \right) \exp\left[\frac{u_A}{F(t)} - 1 - \frac{t}{F(t)} \right] \right\} + 1 \right) \right). \qquad (34)$$

After several simplifications and using Eq. 32 to shorten the expression, it is possible to obtain:

$$d_A(t) = \frac{n\,t + u_A - u(t)}{(n+1)(n+1)} + d_A. \qquad (35)$$

To obtain $d_P(t)$, Equation 35 can be substituted in the $u(t)$ definition expression, resulting in:

$$d_P(t) = \frac{n\,t + u_A + ((n+1)(n+1) - 1)\,u(t)}{(n+1)(n+1)} + d_A. \qquad (36)$$

In the time-domain, Eqs. 32, 35 and 36 assume the following forms, respectively,

$$u(t) = \frac{k\,n(T_A - T_O)(n+1)}{q''(t)} \cdot \left(LambertW\left\{ \left[\left(\frac{q''(t) - q''(t_A)}{q''(t_A)(n+1)} - \frac{n}{(n+1)} \right) \cdot \exp\left(\frac{q''(t) - q''(t_A)}{q''(t_A)(n+1)} - \frac{n}{(n+1)} - \frac{(n+1)\,q''(t) \int_{t_A}^{t} q''(t)\,dt}{r\,l\,k\,n(T_A - T_O)(n+1)} \right) \right] \right\} + 1 \right), \qquad (37)$$

$$d_A(t) = d_A + \frac{n \int_{t_A}^{t} q''(t)\,dt}{r\,l\,(n+1)} + \frac{k\,n(T_A - T_O)}{(n+1)(n+1)\,q''(t_A)} - \frac{k\,n(T_A - T_O)}{(n+1)\,q''(t)} \cdot \left(LambertW\left\{ \left[\left(\frac{q''(t) - q''(t_A)}{q''(t_A)(n+1)} - \frac{n}{(n+1)} \right) \cdot \exp\left(\frac{q''(t) - q''(t_A)}{q''(t_A)(n+1)} - \frac{n}{(n+1)} - \frac{(n+1)\,q''(t) \int_{t_A}^{t} q''(t)\,dt}{r\,l\,k\,n(T_A - T_O)(n+1)} \right) \right] \right\} + 1 \right), \qquad (38)$$

$$d_P(t) = d_A + \frac{n \int_{t_A}^{t} q''(t)\,dt}{r\,l\,(n+1)} + \frac{k\,n(T_A - T_O)}{(n+1)(n+1)\,q''(t_A)} + ((n+1)(n+1) - 1)\frac{k\,n(T_A - T_O)}{(n+1)\,q''(t)} \cdot \left(LambertW\left\{ \left[\left(\frac{q''(t) - q''(t_A)}{q''(t_A)(n+1)} - \frac{n}{(n+1)} \right) \cdot \exp\left(\frac{q''(t) - q''(t_A)}{q''(t_A)(n+1)} - \frac{n}{(n+1)} - \frac{(n+1)\,q''(t) \int_{t_A}^{t} q''(t)\,dt}{r\,l\,k\,n(T_A - T_O)(n+1)} \right) \right] \right\} + 1 \right). \qquad (39)$$

Therefore, for the ablation period, $t \geq t_A$, one can write for the temperature profile:

$$T(x,t) = \begin{cases} T_A, & x \leq d_A(t) \rightarrow \text{ ablated material,} \\ (T_A - T_O)\left(\dfrac{d_P(t) - x}{d_P(t) - d_A(t)} \right)^n + T_O, & x > d_A(t) \text{ and } x < d_P(t), \\ T_O, & x \geq d_P(t). \end{cases} \qquad (40)$$

In conclusion, for the determination of the temperature profile and of the heat penetration depth, Eqs. 18 and 12, in this order, are used for the pre-ablation period. For the ablation period, Eqs. 38, 39 and 40 must be applied for the determination of the ablation depth, the heat penetration depth and the temperature profile.

Results

Constant Heat Flux Case

The analytical model developed in this work was firstly used to solve the ablation problem proposed by Landau[4], for the ablation of Teflon as presented by Blackwell[7] with a constant heat flux case. The material properties and test parameters used are given in Table 1. Blackwell's[7] results were obtained from the graphic presented in the paper using the software SACRID®. The agreement between the present model and Blackwell's[7] results is very good, as can be observed by Figs. 2 and 3. A function of degree 7 ($n = 7$) was assumed for the temperature distribution. Figure 2 shows the temperature profiles for both models (present and Blackwell's[7]), while Fig. 3 shows the percentage error, defined as $(T_{present\ work} - T_{Blackwell})/T_{Blackwell}$, as a function of position and time. The maximum error observed is less than 7 %.

Table 1 – Teflon Thermophysics Properties and Test Parameters ($q'' = 250 Btu/ft^2 s$)

r	$120 lb_m/ft^3$	l	$1000 Btu/lb_m$
k	$3.6 \cdot 10^{-5} Btu/ft\,s\,R$	T_A	$1500\,R$
c_P	$0.3 Btu/lb_m R$	T_O	$536\,R$

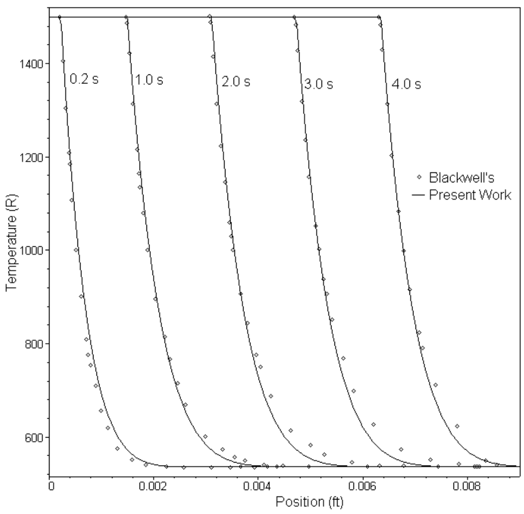

Fig. 2 – Comparison of temperature profiles for the constant heat flux case.

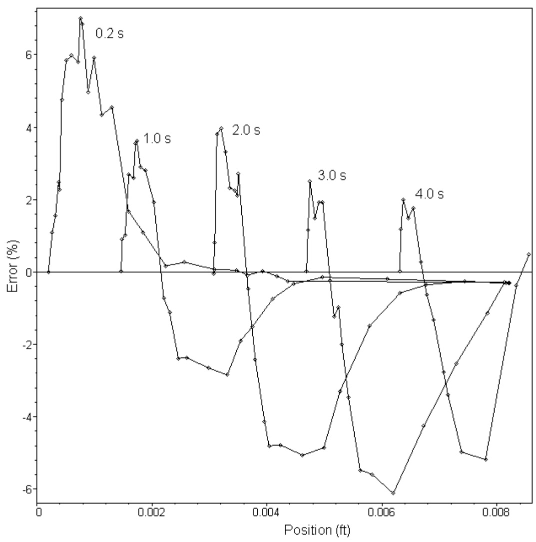

Fig. 3 – Error (%) between the results reported by Blackwell[7] and those in the present work.

Figure 4 presents the heat penetration $d_P(t)$, the ablation $d_A(t)$ and the relative $u(t)$ depths as a function of time. It can be observed that, after a transient period, the system achieves a dynamic equilibrium where the relative depth reaches a constant value. This means that the heat penetration and ablation depths move at the same speed.

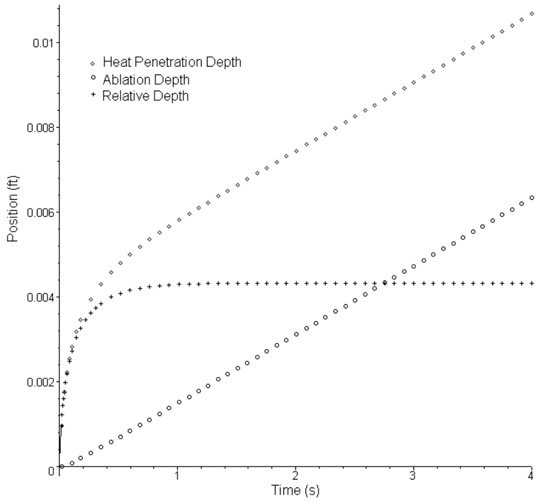

Fig. 4 – Position of $d_P(t)$, $d_A(t)$ and $u(t)$ for the constant heat flux case

Time Variable Heat Flux Case

The theoretical results obtained for the time variable heat flux are compared with literature numerical data in this section. The q-moment scheme as presented by Zien[2] is used in this comparison. This scheme is a refined heat balance method that makes uses of an exponential temperature profiles to develop a set of differential equations that must be numerically solved. This numerical scheme was implemented in a personal computer and the results compared.

It should be observed that Zien's[2] main interest lies on the boundary properties, i.e, on the ablation depth and on ablation speed, rather than on the heat penetration depth or on the temperature profile. So, in order to get a good comparison between the present and Zien's[2] results, the ablation depth equation and its derivative (speed) are made non-dimensional. As already observed, the ablation depth in the present work is obtained from Eq. 38. This expression is divided by its limit (expressed by Eq. 41), to obtain what Zien[2] denominated of ablation position. The ablation speed is given by the derivative of Eq. 38 on time, divided by its limit (expressed by Eq. 42).

$$\lim_{t \to \infty} d_A(t) = \int_{t_A}^{t} \frac{q''(t)}{(n+1)} dt \qquad (41)$$

$$\lim_{t \to \infty} \frac{d\, d_A(t)}{d\, t} = \frac{q''(t)}{(n+1)} \qquad (42)$$

It is important to observe that both models, developed in this work and Zien's[2], present the same ablation position and ablation speed limits (Eq. 41 and 42) which represents the physical situation where the temperature profile has already achieved steady state conditions and all the incoming heat flux is used in the ablation phenomena.

Zien's[2] model was numerically implemented using Maple®V software. Four boundary heat flux cases has been analyzed, using a power-law distribution, as shown in Table 2, for two different values of the Stephan number, which represent two different materials with different capability to change its phase (burn) and to storage the heat inside the virgin material.

Table 2 – Case parameters for comparison with Zien results[2]

Case 1	$q'' = t$	$n = 0.1$
Case 2	$q'' = t$	$n = 1$
Case 3	$q'' = t^3$	$n = 0.1$
Case 4	$q'' = t^3$	$n = 1$

The parameter n used in the theoretical model was computed using the ablation time, t_A, obtained by comparison between the Carslaw and Jaeger[9] equation (Eq. 43) for the surface temperature of a semi-infinite solid subjected to the power-law boundary heat flux, $q'' = F_O\, t^{m/2}$, and Eq. 14, for the heat fluxes cases presented in Table 2. The obtained parameters are presented in Table 3 for the cases described in Table 2. The case 1 and 2, as well as 3 and 4, presents the same ablation time because they have the same boundary heat fluxes.

$$(T_A - T_O) = \frac{F_O}{k}\sqrt{\frac{k}{r\, c_P}}\, \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{3}{2}\right)}\, t_A^{\frac{(m+1)}{2}} \qquad (43)$$

Table 3 – n Parameter and Ablation Time

Case	n Parameter	t_A (Present)	t_A (Zien's[2])
1 / 2	7,589068110	1,208993966 s	1,211413729 s
3 / 4	15,54621919	1,208205637 s	1,208501777 s

Figure 5 shows the ablation speed for cases 1 and 2. From both cases, it can be seen that the present model achieves the ablation speed limits faster than the Zien's[2] data but presents a smaller dumping effect. It's important to note that, with the increase of n, there's an increase in the dumping time for both solutions.

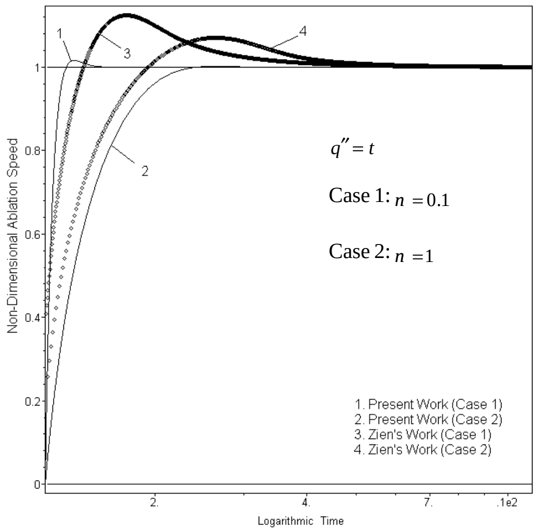

Fig. 5 – Comparison of ablation speed for the cases 1 and 2 between present model and Zien's[2] models.

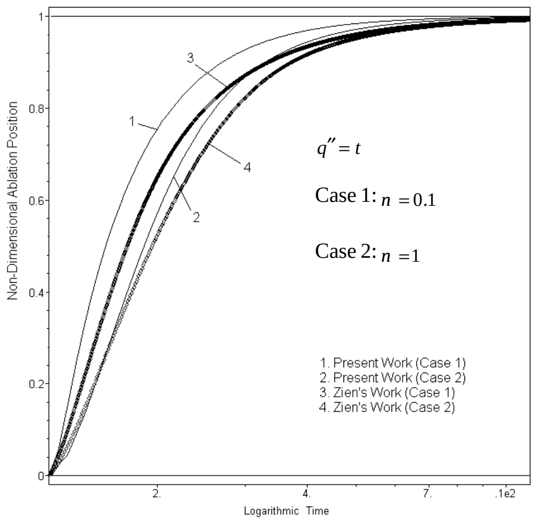

Fig. 6 – Comparison of ablation position for the cases 1 and 2 between present model and Zien's[2] models.

The present work's ablation position shows, at Fig. 6, larger values than the Zien's[2] for any time indicating that the ablation position should be located at an advanced location. Therefore, the use of the theory proposed in this work lead to a conservative result.

In the same way, Figs. 7 and 8 show the ablation speed and ablation position for cases 3 and 4. The same analysis done before is valid . The only new important fact is that the change in the heat flux expression degree has changed the height of the ablation speed curves. It indicates, as expected, that the heat flux degree has a direct influence at the dumping effect.

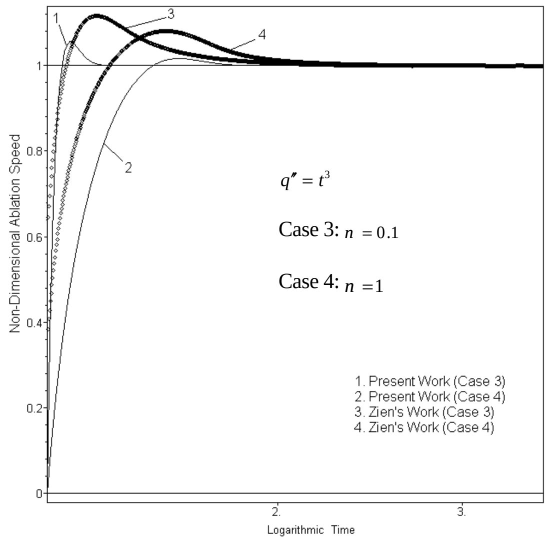

Fig. 7 – Comparison of ablation speed for the cases 3 and 4 between present model and Zien's[2] model.

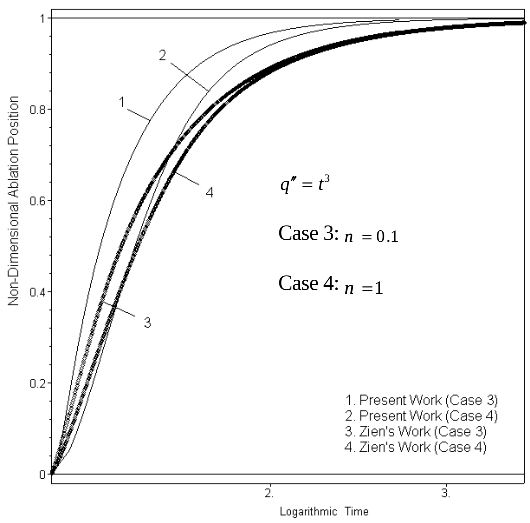

Fig. 8 – Comparison of ablation position for the cases 3 and 4 between present model and Zien's[2] model.

Conclusion

In this paper the ablation problem was solved considering constant material properties and time-variable heat fluxes using the integral method. An approximate analytical closed solution was obtained and compared with numerical solutions presented in the literature for both constant and time variable heat fluxes. The comparison shows good agreement between the behavior of the present and literature solutions. The present results would improve if experimental data were available for comparison. It is interesting to note that the present solution is algebraically closed, *i.e.*, it does not need numerical implementation. Modifications for different types of materials or heat fluxes and can be easily implemented to treat any new situation with constant and time variable heat fluxes.

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