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# APPROXIMATE ANALYTICAL SOLUTION FOR ONE-DIMENSIONAL FINITE ABLATION PROBLEM WITH TIME-CONSTANT HEAT FLUX

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## Abstract

The transient heat transfer in a solid undergoing ablation is a nonlinear problem, which involves a moving boundary that is not known a priori. In this paper the ablation problem is solved with constant material properties and time-constant heat flux using the integral method for a finite one-dimensional solid with a dimensionless formulation. An approximate analytical, closed solution is obtained. The results are compared with solutions presented by the literature.

## Nomenclature

$\delta_P$ Dimensionless Heat Penetration Depth

$\delta_A$ Dimensionless Ablation Depth

$u$ Dimensionless Relative Depth ( $\delta_P - \delta_A$ )

$Q$ Dimensionless Heat Flux ( $Lq'' / k(T_A - T_O)$ )

$\nu$ Inverse Stefan Number ( $\lambda / c_P(T_A - T_O)$ )

$\theta$ Dimensionless Temperature ( $(T - T_O) / (T_A - T_O)$ )

$\tau$ Dimensionless Time ( $\alpha t / L^2$ )

$x$ Dimensionless Length ( $X/L$ )

| | | | |
|---|---|---|---|
| $L$ | Characteristic Length | $t$ | Time |
| $T_A$ | Ablation Temperature | $X$ | Length |
| $T_O$ | Initial Temperature | $\lambda$ | Heat of Ablation |
| $n$ | Function Degree | $k$ | Thermal Conductivity |
| $\alpha$ | Thermal Diffusivity | $q''$ | Heat Flux |

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## Introduction

The ablation problem, as have been treated by many authors, is a complex non-linear phenomenon that can be analyzed from different perspectives, using different techniques and, consequently, presenting different, but similar, results. According to Chung[1] and Zien[2], the exact analytical solution for transient heat transfer in a solid undergoing ablation is very difficult and practically nonexistent even when a simplify model is used. Due to this fact, many numerical, approximate analytical and hybrid solutions have been developed and, almost all of them require considerable numerical computation even for simple cases. As a continuation of a previous work[3], the present one makes use of the integral method[4] to get an approximate closed form analytical solution for the phase-change ablation problem with time-constant heat flux of a one-dimensional finite solid.

## Literature Review

Landau[5] first proposed an idealized ablation problem and solved it for the case of a semi-infinite melting solid with constant properties and with its face heated at constant rate. He applied numerical integration for his solution.

Sunderland and Grosh[6] presented the same problem but described a method of solution using finite differences for the case where the surface is heated by convection. Biot and Agrawal[7] used the variational method for the analysis of ablation with variable properties. Blackwell[8] used the finite volume method with exponential interpolation functions to solve Landau's problem.

Storti[9] considered a one-phase ablation problem as a two-phase Stefan one by the introduction of a fictitious phase occupying the region where the material has been removed. He solved this problem by the finite element method.

Goodman[4] solved Landau's problem using the heat balance integral method with a quadratic temperature profile for a constant heat flux. Zien[2] solved Landau's problem for two specific forms of heat flux with a refined heat balance method using exponential temperature profiles.

## Physical Model

A one-dimensional finite solid with a dimensionless length of one, which is heated at its front surface by a dimensionless time-constant heat flux $Q$, is considered. At the beginning of the process, the heat is conducted inside the material, rising its dimensionless temperature in a region close to the front face, as can be seen at Fig. 1A. The length of this region is called heat penetration depth, $\delta_P$, which varies with time. This period will be called case A. The heating continues until one of the following situations is reached: or the penetration depth reaches the back surface ($\delta_P = 1$, case B), or the front face dimensionless temperature is equal to one unit ($\theta_A = 1$, case C) and starts the ablation phenomenon, or both situations happen at the same time (case D).

The dimensionless time $\tau_B$ is the starting time of case B while $\tau_C$ is the starting time for case C and finally, $\tau_D$ is the starting time for case D.

The back surface is considered insulated then, when the penetration depth reaches the back face, its temperature starts to rise together with the temperature of all the body (Fig.1B). The temperature rise continues until the front surface reaches the dimensionless temperature of one, so that the material presents both conditions of cases B and C at the same time, starting case D.

In case C, the front face dimensionless temperature is equal to one and the surface ablation phenomenon starts. During the ablation, part of the heat is used to keep the surface at the ablating temperature ($\theta_A = 1$) and the remaining heat is used to change the phase of the ablation material. The phase-change event consumes part of the material and the length of the ablated part is denominated ablation depth, $\delta_A$. During case C (Fig. 1C), the material can be divided into three regions: the first, ranging from zero to $\delta_A$, corresponds to the removed material region, the second is the heated material zone, between $\delta_A$ and $\delta_P$, and the third one is from $\delta_P$ to the end, where the material have not felt the presence of the heat flux. The case C ends when the heat penetration front reaches the back surface and, therefore, the case D starts.

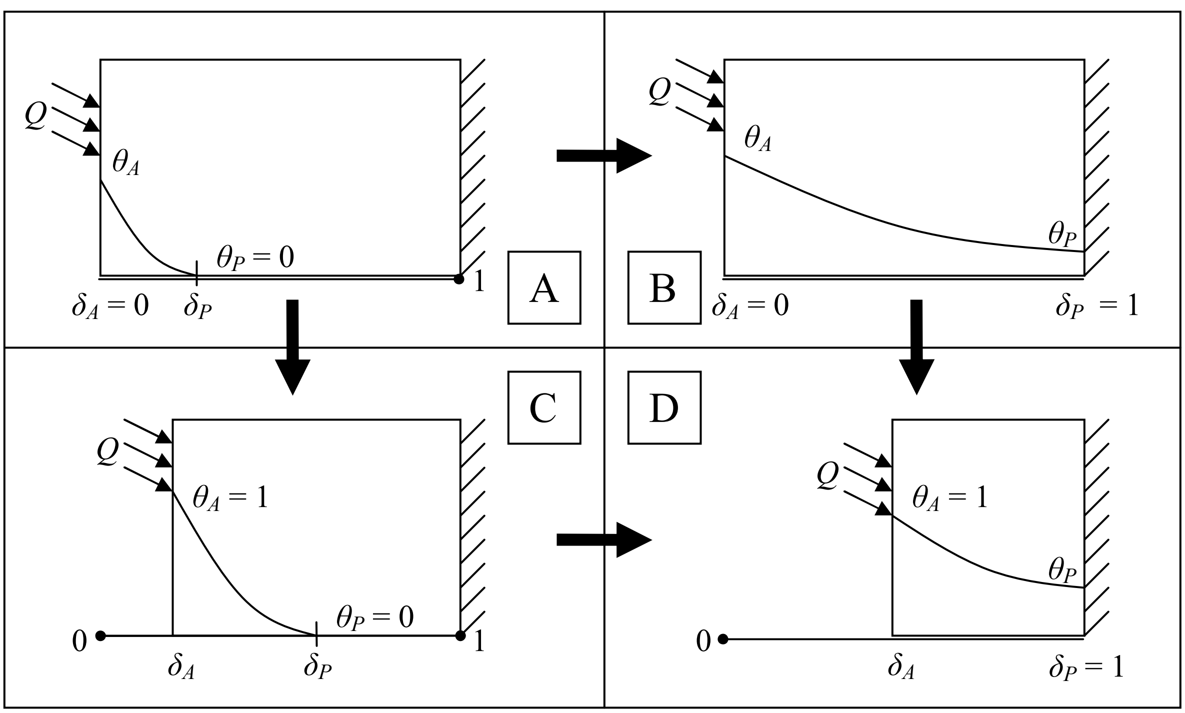


Fig. 1 – Physical model adopted

Figure 1D shows the last case (case D) in which the front surface temperature is equal to one and the heat penetration front has already reached the back surface. In case D, similarly to case C, part of the heat flux is used to heat the material and the rest is consumed in the ablation. In this circumstance, as time goes on, there is less material to heat and the ablation is faster. In the end limit, all material has been ablated.

## Analytical Model

### General Formulation

The following dimensionless one-dimensional transient partial differential heat transfer equation is used to determine the ablation rate and the heat penetration depth:

$$\frac{\partial \theta}{\partial \tau}=\frac{\partial^2 \theta}{\partial x^2} \tag{01}$$

which is valid at $0 \le \tau \le \tau_f$ and $0 \le \delta_A \le x \le \delta_P \le 1$.

The Eq. 1 is integrated in $x$ from $\delta_A$ to $\delta_P$. The results are rearranged using Leibniz's integral formula resulting in:

$$\frac{d}{d\tau}\int_{\delta_A}^{\delta_P}\theta\, dx-\theta_P\frac{d\,\delta_P}{d\tau}+\theta_A\frac{d\,\delta_A}{d\tau}=\left.\frac{\partial \theta}{\partial x}\right|_{x=\delta_P}-\left.\frac{\partial \theta}{\partial x}\right|_{x=\delta_A} \tag{02}$$

To perform this integration, the following temperature profile is considered:

$$\theta = A\frac{(\delta_P - x)^n}{(\delta_P-\delta_A)^n}+B \tag{03}$$

where $A$ and $B$ are time-dependent parameters used to adjust the adopted temperature profile to the problem solution. The $B$ parameter represents the back surface temperature and the $A$ parameter the difference between the front and the back surfaces. This temperature profile is valid in the interval $\delta_A \le x \le \delta_P$.

Substituting Eq. 3 in Eq. 2 and performing the integration, one gets, after some algebra manipulation:

$$\frac{d}{d\tau}\left[\frac{A}{(n+1)}(\delta_P-\delta_A)\right]+(\delta_P-\delta_A)\frac{d\,B}{d\tau}+A\frac{d\,\delta_A}{d\tau}=\frac{A\,n}{(\delta_P-\delta_A)}. \tag{04}$$

### Case A – Pre-Ablation Problem

The following conditions can be considered for the case A:

1. The heat penetration depth is moving, i.e, $\frac{d\,\delta_P}{d\tau}>0$.
2. Ablation front is constant, i.e., $\delta_A=0$.
3. At $x=\delta_P$, $\theta_P=0$ and $-\frac{\partial\theta}{\partial x}=0$.
4. At $x=\delta_A$ the boundary condition are $-\frac{\partial\theta}{\partial x}=Q$ and $\frac{d\,\theta_A}{d\tau}>0$.

From the Eq. 3 and the third boundary condition one gets that

$$B=0, \tag{05}$$

and using the last boundary condition with the Eq. 3 one gets,

$$A=\frac{Q(\delta_P-\delta_A)}{n}. \tag{06}$$

Using these results and the second boundary condition, Eq. 4 can be rewritten as,

$$\frac{d}{d\tau}\left[\frac{Q\delta_P^{\,2}}{n(n+1)}\right]=Q. \tag{07}$$

Integrating, one has:

$$\delta_P=\sqrt{\frac{n(n+1)}{Q}\int_{\tau_A}^{\tau}Q\,d\tau}. \tag{08}$$

Insulating the parameter A, one gets:

$$A=\sqrt{\frac{Q(n+1)}{n}\int_{\tau_A}^{\tau}Q\,d\tau}. \tag{09}$$

Considering a constant heat flux, Eqs. 8 and 9 can be rearranged as,

$$\delta_P=\sqrt{n(n+1)(\tau-\tau_A)} \tag{10}$$

and

$$A=\sqrt{\frac{Q^2(n+1)}{n}(\tau-\tau_A)}. \tag{11}$$

Making $\delta_P(\tau_B)=1$ one can solve Eq. 10 getting

$$\tau_B=\frac{1}{n(n+1)}+\tau_A, \tag{12}$$

On the other hand, considering $A(\tau_C)=1$ and using Eq. 11 one gets

$$\tau_C=\frac{n}{Q^2(n+1)}+\tau_A. \tag{13}$$

Comparing both expressions (Eqs. 12 and 13), if $\tau_B=\tau_C$ one has

$$\frac{1}{n(n+1)}+\tau_A=\frac{n}{Q^2(n+1)}+\tau_A. \tag{14}$$

For $Q=n$, from Eq. 14 it can be shown that $\tau_B=\tau_C$, which represents exactly the transition time between cases A and D. If $Q>n$ then $\frac{n}{Q}<1$ and consequently $\tau_C<\tau_B$ and, in this situation, the problem ranges from case A to case C and later to case D. If $Q<n$ then $\frac{n}{Q}>1$ and $\tau_C>\tau_B$ and the ablation passes from A to B and later to D.

**Case B – Pre-Ablation Problem**

The following conditions can be considered for the case B:

1. The heat penetration depth is stopped at the back surface, i.e, $\delta_P=1$.
2. The ablation depth is stopped at the front surface, where $\delta_A=0$.
3. At $x=\delta_P$ one has $-\frac{\partial\theta}{\partial x}=0$ and $\frac{d\theta_P}{d\tau}>0$.
4. At $x=\delta_A$ the boundary condition are $-\frac{\partial\theta}{\partial x}=Q$ and $\frac{d\theta_A}{d\tau}>0$.

Similarly to what was done for case A, using the temperature profile given by Eq. 3 in the fourth boundary condition, one gets

$$A=\frac{Q(\delta_P-\delta_A)}{n}. \tag{15}$$

Using the first and the second conditions, Eq. 15 can be rewritten as:

$$A=\frac{Q}{n}. \tag{16}$$

From the Eqs. 4, 15 and 16, and the boundary conditions $B_1$ and $B_2$, one can get:

$$\frac{d}{d\tau}\left[\frac{Q}{n(n+1)}\right]+\frac{dB}{d\tau}=Q. \tag{17}$$

Solving Eq. 17 for $B$, one gets:

$$B=\int_{\tau_B}^{\tau}Q\,d\tau-\frac{(Q-Q_B)}{n(n+1)} \tag{18}$$

where $Q_B$ is the heat flux $Q$ at the moment $\tau_B$. At the constant heat flux condition $Q$ one gets

$$B=Q(\tau-\tau_B). \tag{19}$$

$\tau_D$ can be calculated for the instant where the front surface reaches the ablation temperature ($A+B=1$). Considering the heat flux time-variable, it can be seen that:

$$\int_{\tau_B}^{\tau_D}Q\,d\tau=1-\frac{Q_D\,n}{n(n+1)}-\frac{Q_B}{n(n+1)}, \tag{20}$$

or for a constant heat flux

$$\tau_D=\frac{1}{n}\left(\frac{n}{Q}-1\right)+\tau_B. \tag{21}$$

**Case C – Ablation Problem**

The following conditions can be considered for the case C:

1. The heat penetration depth is moving, i.e, $\frac{d\delta_P}{d\tau}>0$.
2. The ablation depth is is moving, i.e, $\frac{d\delta_A}{d\tau}>0$.
3. At $x=\delta_P$ one has $-\frac{\partial\theta}{\partial x}=0$, $\theta_P=0$ and $\frac{d\theta_P}{d\tau}=0$.
4. At $x=\delta_A$ the boundary condition are $-\frac{\partial\theta}{\partial x}=Q-\nu\frac{d\delta_A}{d\tau}$, $\theta_A=1$ and $\frac{d\theta_A}{d\tau}=0$.

Using the temperature profile given by Eq. 3, in the third and forth temperature boundary conditions, the following system is obtained:

$$\begin{cases} \theta_P = A\dfrac{(\delta_P-\delta_P)^n}{(\delta_P-\delta_A)^n}+B=0 \\ \theta_A = A\dfrac{(\delta_P-\delta_A)^n}{(\delta_P-\delta_A)^n}+B=1 \end{cases} \tag{22}$$

Which has the solutions: $B=0$ and $A=1$.

Substituting the expression given by Eq. 3 in the forth (heat flux) boundary condition and solving for $\frac{d\,\delta_A}{d\,\tau}$, one gets after some manipulation:

$$\frac{d\,\delta_A}{d\,\tau}=\frac{Q}{\nu}-\frac{n}{\nu\left(\delta_P-\delta_A\right)}. \tag{23}$$

Substituting Eq. 23, and the results for *A* and *B* in Eq. *4* one can obtain:

$$\frac{d}{d\,\tau}\left[\frac{(\delta_P-\delta_A)}{(n+1)}\right]+\frac{Q}{\nu}-\frac{n}{\nu(\delta_P-\delta_A)}=\frac{n}{(\delta_P-\delta_A)}. \tag{24}$$

With some simple algebra manipulation, this equation can be rewritten as

$$\frac{d}{d\,\tau}(\delta_P-\delta_A)=(n+1)\left[\frac{n+{}^{n}\!/_{\nu}}{(\delta_P-\delta_A)}-\frac{Q}{\nu}\right] \tag{25}$$

Defining *u* as the relative distance between the heat penetration front and the ablation front, i.e, $u=\delta_P-\delta_A$, Eq. 25 can be rearranged as,

$$\frac{d\,u}{d\,\tau}=\frac{(n+1)n(\nu+1)}{u\,\nu}-(n+1)\frac{Q}{\nu} \tag{26}$$

and finally, defining a auxiliary variable, $\gamma=\frac{(n+1)}{\nu}\int_{\tau_C}^{\tau}Q\,d\tau$, and considering *u* as a *γ* dependent function, i.e., $u=u(\gamma)$, Eq. 26 can be expressed as

$$\frac{d\,u}{d\,\gamma}=\frac{n(\nu+1)}{u\,Q}-1\,. \tag{27}$$

The solution of Eq. 27 is:

$$u=\frac{n(\nu+1)}{Q}\cdot\left(LambertW\left\{\left(\frac{Qu_C}{n(\nu+1)}-1\right)\exp\left(\frac{Qu_C}{n(\nu+1)}-1-\frac{Q\gamma}{n(\nu+1)}\right)\right\}+1\right) \tag{28}$$

where $u_C$ is the relative depth at the beginning of case C. Using the original γ variable, the last expression can be rewritten as:

$$u=\frac{n(\nu+1)}{Q}\cdot\left(LambertW\left\{\left(\frac{Qu_C}{n(\nu+1)}-1\right)\exp\left(\frac{Qu_C}{n(\nu+1)}-1-\frac{Q(n+1)}{n(\nu+1)\nu}\int_{\tau_C}^{\tau}Q\,d\tau\right)\right\}+1\right) \tag{29}$$

Substituting Eq. 29 in Eq. 23 one gets,

$$\frac{d\,\delta_A}{d\,\tau}=\frac{Q}{\nu}-\frac{Q}{\nu(\nu+1)\left(LambertW\left\{\left(\frac{Qu_C}{n(\nu+1)}-1\right)\exp\left(\frac{Qu_C}{n(\nu+1)}-1-\frac{Q(n+1)}{n(\nu+1)\nu}\int_{\tau_C}^{\tau}Q\,d\tau\right)\right\}+1\right)} \tag{30}$$

Solving Eq. 30 for $\delta_A$ one obtains:

$$\delta_A=\frac{\int_{\tau_C}^{\tau}Q\,d\tau}{(\nu+1)}+\frac{u_C}{(n+1)(\nu+1)}-\frac{n}{Q(n+1)}\cdot\left(LambertW\left\{\left(\frac{Qu_C}{n(\nu+1)}-1\right)\exp\left(\frac{Qu_C}{n(\nu+1)}-1-\frac{Q(n+1)}{n(\nu+1)\nu}\int_{\tau_C}^{\tau}Q\,d\tau\right)\right\}+1\right) \tag{31}$$

From the *u* definition, one knows that $\delta_P=u+\delta_A$, and using Eqs. 29 and 31, it is given by

$$\delta_P=\frac{\int_{\tau_C}^{\tau}Q\,d\tau}{(\nu+1)}+\frac{u_C}{(n+1)(\nu+1)}+\frac{((n+1)(\nu+1)-1)n}{(n+1)Q}\cdot\left(LambertW\left\{\left(\frac{Qu_C}{n(\nu+1)}-1\right)\exp\left(\frac{Qu_C}{n(\nu+1)}-1-\frac{Q(n+1)}{n(\nu+1)\nu}\int_{\tau_C}^{\tau}Q\,d\tau\right)\right\}+1\right). \tag{32}$$

Considering a constant heat flux and $u_C=\frac{n}{Q}$ and from Eqs. 13 and 10, Eqs. 31 and 32 can be rewritten as

$$\delta_A=\frac{Q(\tau-\tau_C)}{(\nu+1)}+\frac{n}{Q}\frac{1}{(n+1)(\nu+1)}-\frac{n}{Q(n+1)}\cdot\left(LambertW\left\{\left(-\frac{\nu}{(\nu+1)}\right)\exp\left(-\frac{\nu}{(\nu+1)}-\frac{Q^2(n+1)(\tau-\tau_C)}{n(\nu+1)\nu}\right)\right\}+1\right), \tag{33}$$

$$\delta_P=\frac{Q(\tau-\tau_C)}{(\nu+1)}+\frac{n}{Q}\frac{1}{(n+1)(\nu+1)}+\frac{((n+1)(\nu+1)-1)n}{(n+1)Q}\cdot\left(LambertW\left\{\left(-\frac{\nu}{(\nu+1)}\right)\exp\left(-\frac{\nu}{(\nu+1)}-\frac{Q^2(n+1)(\tau-\tau_C)}{n(\nu+1)\nu}\right)\right\}+1\right). \tag{34}$$

The above expression can be used to calculate $\tau_D$ for $\delta_P = 1$, which corresponds to the limit time between case C and case D.

**Case D – Ablation Problem**

The following conditions can be considered for the case D:

1. The heat penetration depth is located at the back surface, i.e, $\delta_P = 1$.
2. The ablation depth moves, $\frac{d\,\delta_A}{d\,\tau} > 0$.
3. At $x = \delta_P$ one has $-\frac{\partial \theta}{\partial x} = 0$ and $\frac{d\,\theta_P}{d\,\tau} > 0$.
4. At $x = \delta_A$ the boundary condition are $-\frac{\partial \theta}{\partial x} = Q - \nu \frac{d\,\delta_A}{d\,\tau}$, $\theta_A = 1$ and $\frac{d\,\theta_A}{d\,\tau} = 0$.

Similarly to Case C, the relative depth, $u$ is used. From Eq. 4 and using the first, a third and fourth boundary conditions, the following ordinary differential equation is obtained:

$$\frac{d}{d\,\tau}[A u] = -(n+1)\frac{A}{u} \quad (35)$$

Substituting Eq.3 in the forth boundary condition and isolating the $\frac{A}{u}$ term, one gets:

$$\frac{A}{u} = \frac{\nu}{n}\frac{d\,u}{d\,\tau} + \frac{Q}{n} \quad (36)$$

Equation 36 can be substituted into Eq. 35 and, after some manipulation, it can be rewritten as:

$$\frac{d}{d\,\tau}\left[\frac{n}{(n+1)}\frac{A u}{\nu} + u + \int_{\tau_D}^{\tau}\frac{Q}{\nu}d\tau\right] = 0 \quad (37)$$

which can be solved for $A$ using the case D initial condition, getting:

$$A = \frac{A_D\,u_D}{u} + \frac{(n+1)\nu}{n\,u}\left(u_D - u - \int_{\tau_D}^{\tau}\frac{Q}{\nu}d\tau\right) \quad (38)$$

where $u_D$ is the relative depth at the $\tau_D$, which, in turn, is the initial dimensionless time for case D. Similarly $A_D$ is the parameter $A$ at $\tau_D$.

Substituting the Eq. 38 at Eq. 36 and isolating the derivative term one can get after some manipulation

$$\frac{d\,u}{d\,\tau} = \frac{u_D\left(\frac{n\,A_D}{\nu} + (n+1)\right) - (n+1)\int_{\tau_D}^{\tau}\frac{Q}{\nu}d\tau}{u^2} - \frac{(n+1)}{u} - \frac{Q}{\nu} \quad (39)$$

Again, similarly to case C, an auxiliary variable, $\eta = \int_{\tau_D}^{\tau}\frac{Q}{\nu}d\tau$, is defined and considering $u$ as a $\eta$ dependent function, i.e., $u = u(\eta)$, Eq. 39 can be expressed as

$$\frac{d\,u}{d\,\eta} = \frac{\nu(n+1)}{Q u^2}\left(\frac{n\,A_D\,u_D}{\nu(n+1)} + u_D - \eta - u\right) - 1 \quad (40)$$

Solving it, through some mathematical manipulation one can gets Eq. 41.

This is a hard to solve transcendental equation. A more friendly expression, based at a physical limits analysis and considering a constant heat flux, can be obtained from Eq. 40:
Considering the instant which $\eta=0$ and, consequently, $u(0)=u_D$, Eq. 40 can be rewritten as

$$\frac{d\,u}{d\,\eta} = \frac{n\,A_D}{Q u_D} - 1 \quad (42)$$

Physically, $\eta_F$ is the instant in which $u=0$, i.e, the auxiliary time when the ablation phenomenon is finished and, therefore, all the income heat flux is used to ablate the solid material.

$$\frac{u\,BesselK\left(0, -\frac{2Q}{(n+1)\nu}\sqrt{\frac{(n+1)\nu}{Q}\left(\frac{A_D u_D n}{(n+1)\nu} + u_D - u - \eta\right)}\right) + \sqrt{\frac{(n+1)\nu}{Q}\left(\frac{A_D u_D n}{(n+1)\nu} + u_D - u - \eta\right)}\,BesselK\left(1, -\frac{2Q}{(n+1)\nu}\sqrt{\frac{(n+1)\nu}{Q}\left(\frac{A_D u_D n}{(n+1)\nu} + u_D - u - \eta\right)}\right)}{u\,BesselI\left(0, \frac{2Q}{(n+1)\nu}\sqrt{\frac{(n+1)\nu}{Q}\left(\frac{A_D u_D n}{(n+1)\nu} + u_D - u - \eta\right)}\right) + \sqrt{\frac{(n+1)\nu}{Q}\left(\frac{A_D u_D n}{(n+1)\nu} + u_D - u - \eta\right)}\,BesselI\left(1, \frac{2Q}{(n+1)\nu}\sqrt{\frac{(n+1)\nu}{Q}\left(\frac{A_D u_D n}{(n+1)\nu} + u_D - u - \eta\right)}\right)}$$
$$= \frac{u_D\,BesselK\left(0, -\frac{2Q}{(n+1)\nu}\sqrt{\frac{A_D u_D n}{Q}}\right) + \sqrt{\frac{A_D u_D n}{Q}}\,BesselK\left(1, -\frac{2Q}{(n+1)\nu}\sqrt{\frac{A_D u_D n}{Q}}\right)}{u_D\,BesselI\left(0, \frac{2Q}{(n+1)\nu}\sqrt{\frac{A_D u_D n}{Q}}\right) + \sqrt{\frac{A_D u_D n}{Q}}\,BesselI\left(1, \frac{2Q}{(n+1)\nu}\sqrt{\frac{A_D u_D n}{Q}}\right)} \quad (41)$$

Numerically it can be shown that

$$\frac{du}{d\eta}=-1. \tag{43}$$

Therefore, one can observe that,

$$\frac{\nu(n+1)}{Qu^2}\left(\frac{n A_D u_D}{\nu(n+1)}+u_D-\eta_F-u\right)=0. \tag{44}$$

Manipulating this equation one gets,

$$\eta_F=\frac{n A_D u_D}{\nu(n+1)}+u_D. \tag{45}$$

The following expression is selected as an approximated solution:

$$u=C\left(1-\frac{\eta}{\eta_F}\right)^m+D\left(1-\frac{\eta}{\eta_F}\right)+E \tag{46}$$

where the terms *C*, *D*, *E* and *m* are obtained though the Eqs. 42, 43 and the boundary conditions $u(0)=u_D$ as well as $u(\eta_F)=0$. Based on these conditions the Eq. 46 is rewritten as:

$$u=(u_D-\eta_F)\left(1-\frac{\eta}{\eta_F}\right)^{\frac{\nu(n+1)\eta_F}{Qu_D^2}}+\eta_F-\eta, \tag{47}$$

and consequently the Eq. 38 becomes,

$$A=-\frac{(n+1)}{n}\frac{\nu\left(\frac{u_D}{\eta_F}-1\right)\left(1-\frac{\eta}{\eta_F}\right)^{\frac{\nu(n+1)\eta_F}{Qu_D^2}}}{\left(\left(\frac{u_D}{\eta_F}-1\right)\left(1-\frac{\eta}{\eta_F}\right)^{\frac{\nu(n+1)\eta_F}{Qu_D^2}}+1-\frac{\eta}{\eta_F}\right)}. \tag{48}$$

Returning to the original variables one gets,

$$u=\frac{n A_D u_D}{\nu(n+1)}\left(1-\left(1-\frac{Q(\tau-\tau_D)}{\frac{n A_D u_D}{(n+1)}+\nu u_D}\right)^{\frac{(n A_D+\nu(n+1))}{Qu_D}}\right)+u_D-\frac{Q}{\nu}(\tau-\tau_D) \tag{49}$$

$$A=\frac{\nu\left(A_D u_D+\frac{(n+1)}{n}(\nu u_D-Q(\tau-\tau_D))\right)}{\left(\frac{n A_D u_D}{(n+1)}\left(1-\left(1-\frac{Q(\tau-\tau_D)}{\frac{n A_D u_D}{(n+1)}+\nu u_D}\right)^{\frac{(n A_D+\nu(n+1))}{Qu_D}}\right)+\nu u_D-Q(\tau-\tau_D)\right)}-\frac{(n+1)\nu}{n} \tag{50}$$

From the *u* definition, $\delta_A=\delta_P-u$, and it is given by

$$\delta_A=1-\frac{n A_D u_D}{\nu(n+1)}\left(1-\left(1-\frac{Q(\tau-\tau_D)}{\frac{n A_D u_D}{(n+1)}+\nu u_D}\right)^{\frac{(n A_D+\nu(n+1))}{Qu_D}}\right)-u_D+\frac{Q}{\nu}(\tau-\tau_D). \tag{51}$$

In a similar way using Eq. 50 and the temperature statement of the forth boundary condition, one can demonstrate that

$$B=1-\frac{\nu\left(A_D u_D+\frac{(n+1)}{n}(\nu u_D-Q(\tau-\tau_D))\right)}{\left(\frac{n A_D u_D}{(n+1)}\left(1-\left(1-\frac{Q(\tau-\tau_D)}{\frac{n A_D u_D}{(n+1)}+\nu u_D}\right)^{\frac{(n A_D+\nu(n+1))}{Qu_D}}\right)+\nu u_D-Q(\tau-\tau_D)\right)}+\frac{(n+1)\nu}{n} \tag{52}$$

Figure 2 presents a plot of the parameter u as a function of η using Eq. 46 and Eq. 41, as well as the results obtained through numerical calculation of Eqs. 40 using the algebra manipulation software Maple®.

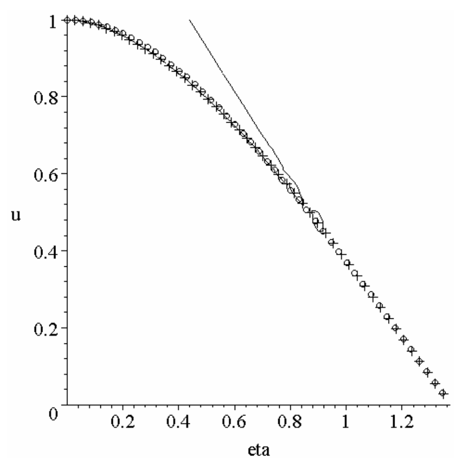


Fig. 2 – Comparison between different solution of Eq. 40 (Parameters used: Q=3 n=3 ν=2, $u_D$=1, $\eta_F$=11/8, $A_D$=1)

From this plot, it can be observed that the values obtained by means of the numerical solution methods applied to the exact solution (Eq. 40, cross) agree very well with the approximated analytical solution (Eq. 47, circle). On the other side, the complicated exact solution given by Eq. 41, continuous line present so many numerical errors during the evaluation of that expression, that the result expressed by Eq. 41 is useless for practical purposes.

## Results

In this section, results obtained for four different heat flux conditions are presented. At all graphics the following notation is used: the blue line indicates the temperature difference between the front and the back surface, which is evaluated by the *A* parameter at the temperature profile; the green line is the back surface temperature, evaluated by the *B* parameter of the temperature profile; the black line is the ablation front $\delta_A$ as presented at the physical model; the red line is the *u* variable which represents the difference between the heat penetration and the ablation fronts; and the orange line represents the heat penetration front. All the x-axis corresponds to the dimensionless time and the y-axis to the normalized dimensionless variable of interest.

The case for which $Q<n$ is presented in Fig. 3. It can be noted that the temperature difference between the front and back surface (blue curve) remains constant during the case B time period and that the ablation starts when this difference starts to reduce. In case D end of the time period range, the ablation speed is the highest of the process.

Figure 4 shows the case where the dimensionless heat flux *Q* is equal to the *n* degree used at the temperature profile. In this situation, the system goes from case A directly to case D, which can be noted by the non-existence of any constant value level for the temperature differences or for the *u* variable.

Another interesting case is when the heat flux is greater than *n* and the system goes from case A to case D passing through case C. These cases are present at Fig. 5 for $Q$=5 and Fig. 6 for $Q$=8. It is interesting to observe that with the increasing of the heat flux there is a change at the concavity of the temperatures curves (blue, *A* parameter, and green, *B* parameter). This behavior occurs because, with the increase of the heat flux, the *B* parameter presents an increase of its velocity at the end of case D, as can be observed with Eq. 52.

In a general way, the results presented at Figs. 3-6 can be considered mathematically correct and physically consistent, as demonstrated along this paper. Figure 7 shows a plot for the present model, similar to Figs 3 to 6, and the results obtained from Chung[1] work using the Finite Difference Method for the same problem presented at this work. The Chung's solution was implemented for comparison with present results.

At Fig. 7 the front surface temperature is represented with the blue line and the back surface with the red one. The difference between the temperatures (front and back surface) is shown at the black line and the ablation front position with the green line. The thinner lines are used to the present work solutions and the thickest to the Chung's ones.

Based at Fig. 7 one can observed that the developed solution presents the same trends of the Chung's results, but there are considerable differences between the results of the solutions methods used. Actually, this disagreement between models can be easily explained by the fact that, in the present work, it is not considered a transient behavior at the transition between the cases considered. Therefore, the shift between results starts in the transition time between Cases A and B, where a difference in the initial derivative causes all the discrepancy of the absolute values of position and temperatures observed for all the time range studied.

It is recommended that in the near future a transient behavior between cases should be adopted, or using the blending technique (see Yovanovich[10]) or adopting other boundary conditions.

## Conclusions

In the present work, an approximate analytical solution was developed using the integral method to the transient phase-change ablation problem with time-constant heat flux applied over a surface of a one-dimensional finite solid. The results were not conclusive but they indicate a good perspective of development for the near future. The goals for the continuation of the present work are to get a better comparison with results from the literature as well as to obtain data from experimental studies to be performed at the laboratory.

## Figures

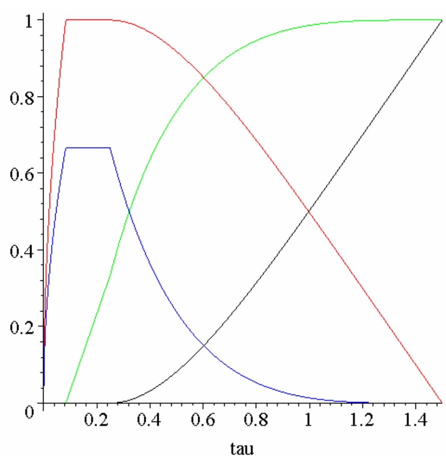


Fig. 3 – Parameters used: Q=2 n=3 ν=2

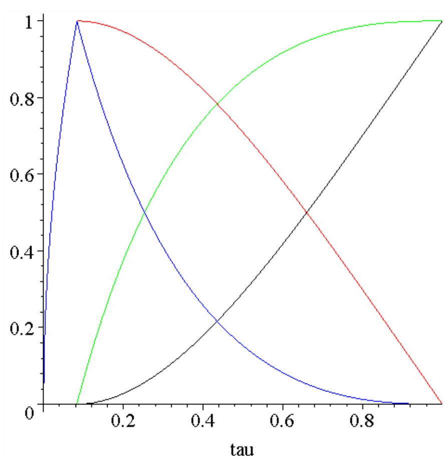


Fig. 4 – Parameters used: Q=3 n=3 ν=2

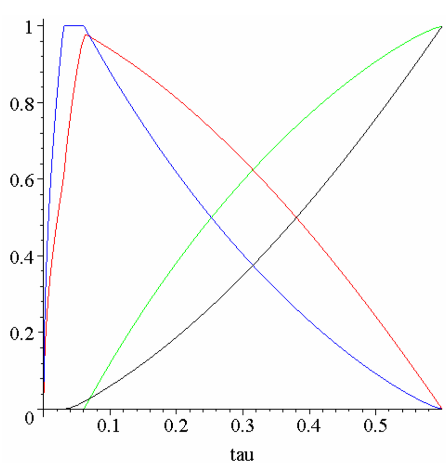


Fig. 5 – Parameters used: Q=5 n=3 ν=2

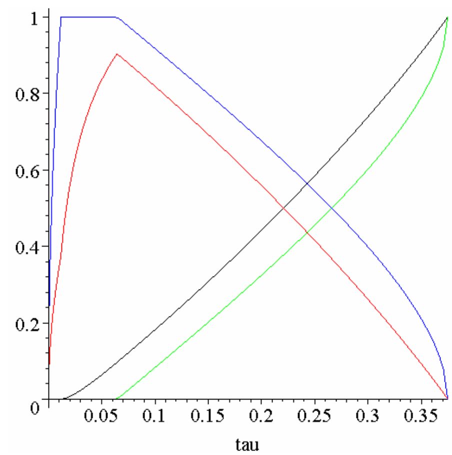


Fig. 6 – Parameters used: Q=8 n=3 ν=2

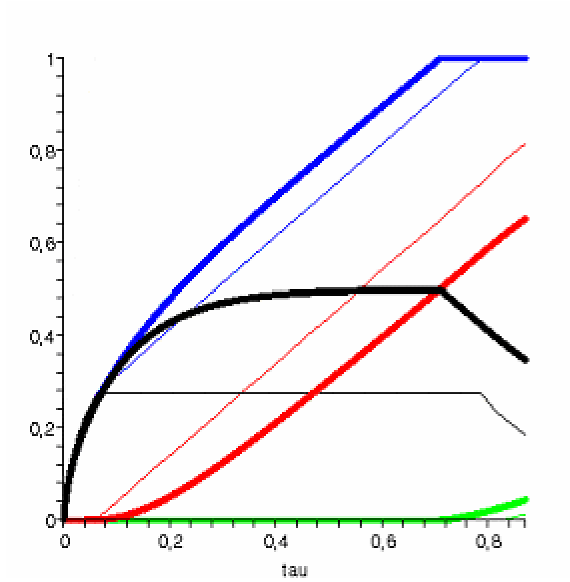


Fig. 7 – Comparison between Chung's solution[1] and present work.

(Parameters used: Q = 1, n = π / (4 – π), ν = 1)

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