A NEW APPROACH FOR THE HEAT BALANCE INTEGRAL METHOD APPLIED TO HEAT CONDUCTION PROBLEMS

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In the present paper, the heat conduction problem for the semi-infinite solid, with prescribed temperature and heat flux boundary conditions is solved. The Heat Balance Integral Method, based on an *n* exponent temperature profile is applied. One of the main concerns of this work is the determination of the parameter *n*, obtained using an energy balance in the solid. The literature classical solutions (Carslaw and Jaeger, 1959), were used in the comparison with temperature distributions obtained for several different values of *n* profiles, showing that the parameters selected in the present paper are the best ones. Different values were obtained for the prescribed heat flux and for the prescribed temperature boundary condition problems. The results obtained are useful for the design of the thermal protection systems in reentry space vehicles.

Nomenclature

T	=	Temperature [K]	
T_O	=	Initial temperature [K]	
T_R	=	Reference temperature [K]	
T_F	=	Prescribed temperature [K]	
t	=	Time [s]	
ρ	=	Density [kg/m^3]	
c_P	=	Heat capacity [J / kg K]	
k	=	Conductivity [W / m K]	
X	=	Length [m]	
Δ_P	=	Heat penetration depth [m]	
L	=	Arbitrary length [m]	
q''_F	=	Prescribed heat flux [W / m^2]	
n	=	Exponent of the temperature profile	
A	=	Time-dependent parameter	
α	=	Heat diffusivity [m^2 / s]	$(k / \rho\, c_P)$
δ_P	=	Non dimensional heat penetration depth	(Δ_P / L)
θ	=	Non dimensional temperature	$(T - T_O) / (T_R - T_O)$
x	=	Non dimensional length	(X / L)
τ	=	Non dimensional Fourier time	$(\alpha\, t) / L^2$
θ_F	=	Non dimensional prescribed temperature	$(T_F - T_O) / (T_R - T_O)$
θ_S	=	Non dimensional surface temperature	$(T_S - T_O) / (T_R - T_O)$
Q_F	=	Non dimensional prescribed heat flux	$(q''_F\, L / k\, (T_R - T_O))$
η	=	Non dimensional auxiliary variable	$(x / \tau^{1/2})$

I. Introduction

Conduction heat transfer is a very important phenomenon to the engineering science. Since the Fourier's work "La Théorie Analytique de la Chaleur", many mathematics methods has been developed to assist modern engineers, helping them to understand and predict the thermal behavior of different problems. Even being one of

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the most studied subjects on the thermal engineering field, heat conduction is still an interesting phenomenon with many challenging problems that still need to be solved. Some examples of these unsolved problems can be found within a material undergoing ablation, due to an imposed heat flux or temperature in its external free surface, as it happens in the thermal protection materials of reentry spacecrafts and satellites.

The authors of the present paper have being studying analytically the heat conduction and ablation problems using the electrical analogy (Braga et al. 2002)[1] and the Heat Balance Integral Method (HBIM). Braga et al. (2003)[2] studied the semi-infinite conduction problem, using the HBIM, subjected to the prescribed time-variable heat flux boundary condition in its free surface. This same authors (Braga et al., 2004)[3] studied this same problem, subjected to the free surface time-constant heat flux boundary condition in a finite solid with an insulated opposite surface. In both papers, the pre-ablation and ablation phases were considered. From these previous works, it was observed the need to understand better the effect of the selected temperature profile used at the HBIM on the accuracy of the obtained solution.

As it will be described later, in all these problems the solid is subjected to pure conduction heating process, before undergoing the ablation. Table 1 shows the combination of some possible heat conduction problems for the pre-ablation period, according to the boundary conditions applied. In this Table, the left two columns represent the boundary conditions applied to the right edge of the solid, which can be considered semi-infinite or finite. For the finite case, the following boundary conditions can be considered: prescribed temperature (PT), prescribed heat flux (PH) and convection (C). The three right columns of Table 1 represent the left free surface, which will be ablated after the appropriate temperature level is achieved (the ablation will not be considered in the present paper). The left surface can, in turn, be subjected to the same boundary conditions. All the possible combinations can be found in this table. Only the cases marked with an *X* in the table are considered in the present work, but the methodology presented can be used to solve all the remaining cases. Figure 1 illustrates the heat conduction problem for the semi infinite and finite solid with the three mentioned left and right boundary conditions (PT, PH and C).

The Heat Balance Integral Method (HBIM), as presented by Goodman (1964)[4], is used in the analytical solution of the present cases. In this method, an *n* exponent temperature profile is assumed for the temperature distribution. The precision of the method is directly related to the correct choice of this parameter. In the present paper, a procedure used to select the appropriate *n* parameter is presented.

Analyzing the thermal behavior of a heated solid, it can be noted that, in the pre-ablation problems, at the very first moment, the temperature profile is the same of that obtained for a simple problem without ablation. As the time goes on, the temperature increases, starting the ablation. As a consequence, the temperature profile changes and, in the limit, as time goes to infinity, it tends to that observed for a surface prescribed temperature (without ablation) problem. Therefore, one can conclude that a good choice of the profile exponent *n* should be a time dependent parameter, able to capture the transition between these limiting curves.

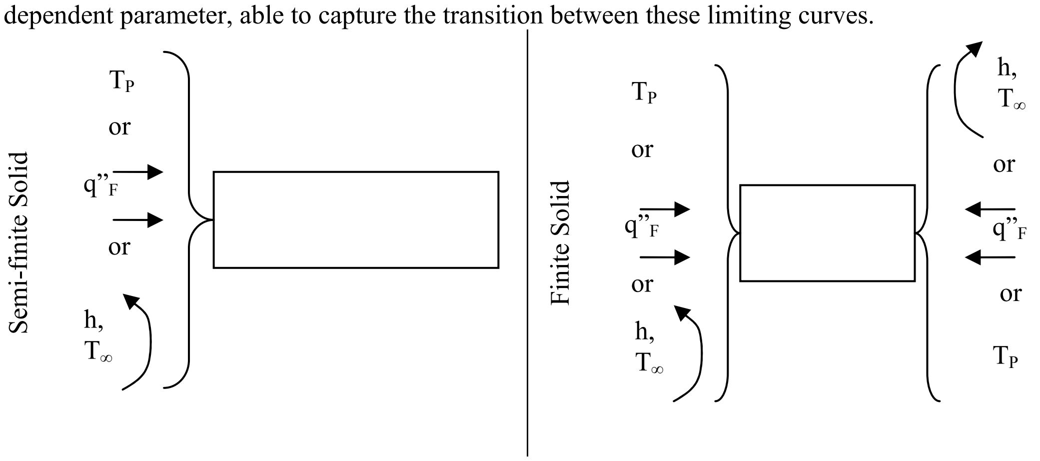

Figure 1. Graphic representation of the possible heat conductions problems.

Table 1. Possible pre-ablation problems according to the applied boundary conditions.

		Left free surface boundary condition		
		PT	PH	C
Right boundary condition	Semi-infinite	X	X	
	Finite – PT			
	Finite – PH			
	Finite - C			

In the present paper, the Heat Balance Integral Method is used to solve the heat conduction problems for a semi-infinite solid without ablation using a time variable exponent temperature profile *n*. It is important to observe that, as far as present authors know, no work regarding the use a not integer time variable value of *n* for the HBIM is available in the literature.

II. Physical Modeling

The case of a one dimensional semi-infinite solid body made of an isotropic material with constant properties is considered. The body is assumed to be at a constant temperature until the start up of the heating process. Two different heating conditions are considered in this paper. The first one is a prescribed constant heat flux (Q_F) at the free (left) surface of the material; the other one is a prescribed constant temperature (θ_F). At both processes the heat is conducted inside the material generating two different sections: the heated region, which the temperature is affected by the heating imposed at the surface of the material and the virgin region, where the material is still not affected by the heating, remaining at the initial temperature. The distance between the free left surface of the material and the front end of these regions is named as the heat penetration depth (δ_P). Figure 2 illustrates the physical model scheme adopted for the prescribed heat flux case.

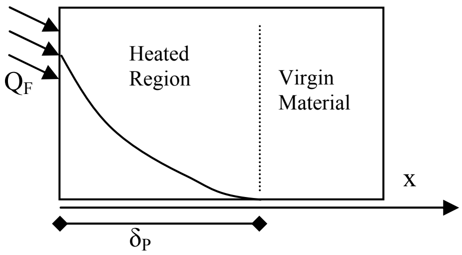

Figure 2. Physical modeling scheme, for the prescribed heat flux case.

III. Mathematical Modeling

In this section, the mathematical models used to predict the thermal behavior of the problems considered are developed.

A heat balance over the body (see Fig. 2) leads to the following well known transient non-dimensional differential heat equation, where θ=(T –T_O)/(T_R –T_O), T_O is the initial temperature and T_R is any arbitrary reference temperature:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^2}\ . \tag{1}$$

In this equation, τ is the Fourier non dimensional time, defined as $\tau = (\alpha\ t)/L^2$, where α is the heat diffusivity, t is the time and L corresponds to an arbitrary length. Also, x is defined as $x = X / L$, where X is a dimensional length.

The following boundary conditions are considered for these problems. If there is not a virgin region in the material, i.e., if all the solid material already experiences the presence of the heating imposed at the surface, the boundary condition considered for the right edge located at the infinity is:

$$\lim_{x\to\infty}\theta = 0 \ \text{ and } \ \lim_{x\to\infty}\frac{\partial \theta}{\partial x} = 0\ . \tag{2}$$

If there is a virgin region, i.e., if the time is not enough for the heat to reach the entire solid, the boundary conditions, located in the border line between the heated and the virgin regions, are:

$$\theta\big|_{x=\delta_P} = 0 \ \text{ and } \ \left.\frac{\partial \theta}{\partial x}\right|_{x=\delta_P} = 0\ . \tag{3}$$

At the free surface of the material, two boundary conditions can be found, depending on the heating method used. For the prescribed heat flux case, the following equation is used:

$$-\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = Q_F \quad . \tag{4}$$

On the other hand, for the prescribed temperature heating method, the expression is:

$$\left.\theta\right|_{x=0} = \theta_F \quad . \tag{5}$$

IV. Solutions

In this section, the problem under investigation will be solved, using two different analytical methods.

A. Classical Solution

The classical solutions found in the literature are obtained using the Laplace Transform technique or using a variable transformation, as presented by many of the classical conduction heat transfer books, such as Arpaci (1966)[5], Carslaw and Jaeger (1959)[6], among others. These solutions are reproduced for the two problems under analysis, as follows.

1. Prescribed Heat Flux Problem

For the constant heat flux at the free surface case, the solution of the partial differential equation (Eq. 1) subjected to the boundary conditions given by Eqs. 2 and 4 is:

$$\theta = 2Q_F\sqrt{\tau}\left\{\frac{e^{\frac{-x^2}{4\tau}}}{\sqrt{\pi}} - \frac{x}{2\sqrt{\tau}}\,erfc\left(\frac{x}{2\sqrt{\tau}}\right)\right\} \quad . \tag{6}$$

The Duhamel's theorem needs to be used to obtain the solution for a variable heat flux solution boundary condition on the free surface, when needed.

2. Prescribed Temperature Problem

For the prescribed temperature problem, the classical solution for Eq. 1 subjected to the Eqs. 2 and 5 boundary conditions, is obtained with the use of the Laplace transform technique or of a proper variable substitution. The following solution is obtained for the time-constant prescribed temperature case:

$$\theta = \theta_F\, erfc\left(\frac{x}{2\sqrt{\tau}}\right). \tag{7}$$

Again, the solution for a prescribed time-variable temperature boundary condition problem is obtained with the use of the Duhamel's theorem.

B. Heat Balance Integral Method

The Heat Balance Integral Method (HBIM) is based on the integral form of the heat conduction differential equation. This form is obtained by the integration of Eq. 1 with respect at the position *x,* from the solid surface ($x=0$) up to the heat penetration depth, ($x=\delta_P$). Doing so, the following equation is obtained:

$$\int_0^{\delta_P}\frac{\partial\theta}{\partial\tau}dx = \left.\frac{\partial\theta}{\partial x}\right|_{x=\delta_P} - \left.\frac{\partial\theta}{\partial x}\right|_{x=0} . \tag{8}$$

As δ_P is a time-dependent variable, the Leibniz rule is used and the Eq. 8 is rearranged as:

$$\frac{d}{d\tau}\left[\int_0^{\delta_P}\theta\, dx\right] - \theta\Big|_{x=\delta_P}\frac{d\,\delta_P}{d\tau} = \frac{\partial\theta}{\partial x}\Big|_{x=\delta_P} - \frac{\partial\theta}{\partial x}\Big|_{x=0} . \tag{9}$$

At this point, an appropriate function has to be selected as the profile of the temperature distribution inside the material. This function must have a good agreement with the space boundary conditions and must present time dependent parameters, which are determined using the remaining time boundary conditions. In this paper, the following profile is considered:

$$\theta = A\left(\frac{\delta_P - x}{\delta_P}\right)^n , \tag{10}$$

where A is the time-dependent parameter, calculated through the free surface boundary condition. Physically, this parameter represents the surface temperature of the material, while n is the exponent of the Eq. 10 and establishes the shape of the temperature profile along the solid, being arbitrarily selected. The best selection of the n value and its implications will be explained latter on this paper. The profile represented by Eq. 10 naturally satisfies the boundary conditions given by Eq. 3. Substituting Eq. 10 in Eq. 9 one gets the following ordinary differential equation:

$$\frac{d}{d\tau}\left[\frac{A\,\delta_P}{(n+1)}\right] = \frac{A\,n}{\delta_P} . \tag{11}$$

1. *Prescribed Heat Flux Problem*

The prescribed heat flux problem is solved in this section. The temperature distribution (Eq. 10) is substituted in the heat flux boundary condition (Eq. 4), obtaining the following equation:

$$\frac{A\,n}{\delta_P} = Q_F . \tag{12}$$

Solving for A, one gets:

$$A = \frac{Q_F\,\delta_P}{n} . \tag{13}$$

Substituting the Eq. 13 in Eq. 11 the following differential equation is obtained:

$$\frac{d}{d\tau}\left[\frac{Q_F\,\delta_P^{\,2}}{n(n+1)}\right] = Q_F . \tag{14}$$

This equation can be easily solved for the heat penetration depth, resulting in:

$$\delta_P = \sqrt{\frac{n(n+1)}{Q_F}\int_0^{\tau} Q_F\, d\tau} . \tag{15}$$

Note that both parameters Q_F and n are considered time dependent in this solution.

Summarizing, for the prescribed heat flux case, the temperature profile is given by Eq. 10, the surface temperature by Eq. 13 and the heat penetration depth (δ_P) by Eq. 15. Therefore, the only unknown variable is the n parameter.

2. *Prescribed Temperature Problem*

Similarly to the procedure adopted in the last section, the prescribed temperature problem is solved by substituting the temperature distribution given by Eq. 10 in the prescribed temperature boundary condition expression, presented in Eq. 5, obtaining the following expression:

$$A = \theta_F \quad . \tag{16}$$

Substituting Eq. 16 in Eq. 11 the following differential equation is obtained:

$$\frac{d}{d\,\tau}\left[\frac{\theta_F\,\delta_P}{(n+1)}\right] = \frac{\theta_F\,n}{\delta_P} \quad . \tag{17}$$

This equation can be solved for the heat penetration depth. The prescribed temperature and the n parameter are considered time-dependent. The following solution is found:

$$\delta_P = \sqrt{\frac{2\,(n+1)^2}{\theta_F{}^2}\int_0^\tau \frac{n}{(n+1)}\theta_F{}^2\,d\tau} \quad . \tag{18}$$

Summarizing, the temperature profile is given by Eq. 10, the A parameter by the Eq. 16 and the heat penetration depth (δ_P) by Eq. 18. The only unknown is the value of the n parameter.

C. Comparison between the solutions obtained through the Heat Balance Integral Method

For the two analyzed cases, the same temperature profile (Eq. 10) was used as the starting point for the HBIM. Due to this, both solutions can be compared. These solutions are presented in Table 2 where the two last lines are included for the analysis of a case where some simplifications are added to the problem.

It is interesting to observe that the heat penetration depth solution presents similar mathematical expressions, where two terms can be recognized. The first one is an inertia term, which is expressed by an integral divided by its integration term, reproduced here for both the free surface prescribed heat flux and temperature, respectively: $\int_0^\tau Q_F\,d\tau \Big/ Q_F$ and $\int_0^\tau \theta_F{}^2\,d\tau \Big/ \theta_F{}^2$. This terms acts as a "dumping" of the variable heat flux or temperature imposed in the free surface, decreasing the velocity of the heat penetration front, while keeping the boundary condition influence on the solution. The second term in the heat penetration depth expressions presents the level of the solution dependence on the selected temperature profile, being very similar for both cases. One should note that, with the exception of the surface temperature at the prescribed temperature case, all solutions depend very much of the temperature profile exponent, the n parameter. It should be emphasized again that the precise determination of this parameter is the main key to minimize the approximation errors found in the HBIM.

Table 2. Summary of the HBIM solutions.

	Prescribed Heat Flux	Prescribed Temperature
Surface Temperature	$A = \frac{Q_F\,\delta_P}{n}$	$A = \theta_F$
Heat Penetration Depth	$\delta_P = \sqrt{\frac{n(n+1)}{Q_F}\int_0^\tau Q_F\,d\tau}$	$\delta_P = \sqrt{\frac{2\,(n+1)^2}{\theta_F{}^2}\int_0^\tau \frac{n}{(n+1)}\theta_F{}^2\,d\tau}$
Heat Penetration Depth (constant n value)	$\delta_P = \sqrt{\frac{n(n+1)}{Q_F}\int_0^\tau Q_F\,d\tau}$	$\delta_P = \sqrt{\frac{2\,n(n+1)}{\theta_F{}^2}\int_0^\tau \theta_F{}^2\,d\tau}$
Heat Penetration Depth (constant n, Q_F and θ_F values)	$\delta_P = \sqrt{n(n+1)\tau}$	$\delta_P = \sqrt{2\,n(n+1)\tau}$

V. Obtaining the parameter *n* – Energy Balance

As described before, the n parameter is the exponent of the temperature distribution profile adopted and, generally, is arbitrarily selected. Usually, this profile has an exponential function shape, which satisfies the Eq. 2 boundary conditions, or a polinomium, that instead, satisfies the boundary conditions expressed by Eq. 3. In the present work, the n value is free to attain any real decimal value and to vary with time. Goodman (1964)[4] presents the error (difference between classical and HIBM solutions, divided by classical solution) for some polinomium with different degrees. This author showed that the surface temperature for the prescribed heat flux condition presents large error for $n = 2$ (*8,6%*), which decreases for $n = 3$ (*2,0%*) but there is no guarantee that increasing the order of the polynomial will improve the accuracy.

To find out the best value for n, the classical solution, which is exact, can be used. The strategy adopted is to perform an energy balance in the solid, using both the classical and the HBIM solutions. The total energy in these cases should be the same.

To simplify the calculations, a non-dimensional variable, defined as $\eta = x/\tau^{1/2}$, is adopted in this study. The classical solutions (Eqs. 6 and 7), in terms of this new variable, for the prescribed heat flux and prescribed temperature are, respectively:

$$\theta = \frac{2Q_F\sqrt{\tau}}{\sqrt{\pi}}\left\{e^{\frac{-\eta^2}{4}} - \frac{\sqrt{\pi}}{2}\eta\, erfc\left(\frac{\eta}{2}\right)\right\}, \tag{19}$$

$$\theta = \theta_F\, erfc\left(\frac{\eta}{2}\right). \tag{20}$$

For the HBIM solutions, one can obtain the following temperature profile, for the prescribed time-constant heat flux problem:

$$\theta = Q_F\sqrt{\tau}\sqrt{\frac{(n+1)}{n}}\left(1 - \frac{\eta}{\sqrt{n(n+1)}}\right)^n. \tag{21}$$

For the prescribed time-constant temperature, the profile is given by:

$$\theta = \theta_F\left(1 - \frac{\eta}{\sqrt{2n(n+1)}}\right)^n. \tag{22}$$

Normalizing the expressions with respect to the surface temperature (θ_S for $\eta = 0$) one gets, for Eqs. 19 and 20:

$$\frac{\theta}{\theta_S} = \left\{e^{\frac{-\eta^2}{4}} - \frac{\sqrt{\pi}}{2}\eta\, erfc\left(\frac{\eta}{2}\right)\right\}, \tag{23}$$

$$\frac{\theta}{\theta_S} = erfc\left(\frac{\eta}{2}\right). \tag{24}$$

Similarly, the Eqs. 20 and 21 can be rewritten as:

$$\frac{\theta}{\theta_S} = \left(1 - \frac{\eta}{\sqrt{n(n+1)}}\right)^n, \tag{25}$$

$$\frac{\theta}{\theta_S} = \left(1 - \frac{\eta}{\sqrt{2n(n+1)}}\right)^n. \tag{26}$$

Now, integrating Eqs. 23 and 24 over η from 0 to ∞ one gets, respectively:

$$\int_0^\infty \frac{\theta}{\theta_S} d\eta = \frac{\sqrt{\pi}}{2}\ , \tag{27}$$

$$\int_0^\infty \frac{\theta}{\theta_S} d\eta = \frac{2}{\sqrt{\pi}}\ . \tag{28}$$

On the other hand, integrating Eq. 25 over η from *0* to $(n(n+1))^{1/2}$ one gets:

$$\int_0^{\sqrt{n(n+1)}} \frac{\theta}{\theta_S} d\eta = \sqrt{\frac{n}{(n+1)}}\ . \tag{29}$$

Finally, integrating Eq. 26 over η from *0* to $(2n(n+1))^{1/2}$ the following expression is obtained:

$$\int_0^{\sqrt{2n(n+1)}} \frac{\theta}{\theta_S} d\eta = \sqrt{\frac{2n}{(n+1)}}\ . \tag{30}$$

Considering that the energy balance obtained from the classical and HBIM solutions should be the same for the prescribed heat flux problem, the Eq. 27 right hand side must be equal to the right hand side of Eq. 29, obtaining the following equation:

$$\frac{\sqrt{\pi}}{2} = \sqrt{\frac{n}{(n+1)}}\ , \tag{31}$$

which can be solved for n. The solution is:

$$n = \frac{\pi}{(4-\pi)} \approx 3{,}659792369\ . \tag{32}$$

Similarly, for the prescribed temperature problem, the right hand side of Eq. 28 has to be equal to the right hand side of Eq. 30 leading to the equation:

$$\frac{2}{\sqrt{\pi}} = \sqrt{\frac{2n}{(n+1)}}\ , \tag{33}$$

which solution is:

$$n = \frac{2}{(\pi-2)} \approx 1{,}751938393\ . \tag{34}$$

Eqs. 32 and 34 n values, which were obtained based on an energy balance, are proposed as the best values for the exponent of the temperature profile (Eq. 10) used at the HBIM for each specific case.

VI. Comparison between the Classical Solution and the Heat Balance Integral Method

The comparison between the classical and the HBIM solutions is made through the temperature profiles (Eqs. 19 to 22) and through the surface temperature for the prescribed heat flux problem.

A. Prescribed Heat Flux

Figure 3 presents the plots of the normalized temperature with respect to the surface temperature (θ / θ_S) as a function of the parameter η. The full line represents the graphic of the classical solution (given by Eq. 19) and the points those for the HIBM solutions (given by Eq. 21), considering different values of the parameter n (2, 3, 4 and that given by Eq. 32). In Fig. 4, the absolute error for the HBIM (defined as ε_{abs} = *Eq. 23 – Eq. 25*) is presented for different n values. The maximum positive and negative absolute errors are also tabulated in Table 3.

It can be noted in Fig. 3 that the temperature distribution curves for $n = 4$, $n = 3.66$ and the classical solution present very similar shapes, being hard to distinguish among them. In Fig. 4, the absolute errors curves for the cases $n = 4$ and $n = 3.66$ have similar behavior, but the curve for $n = 3.66$ shows the minimum error levels, being the best choice among all the cases studied, as it can also be observed in the values presented in Table 4. Actually for larger values of η, the $n = 4$ profile seems to be the better choice, but if one takes the area between the error curve and the η axis, one can see that the $n = 3.66$ curve positive area is almost the same as the negative one, showing that this profile in average represents better the actual curve, being the best profile to be selected.

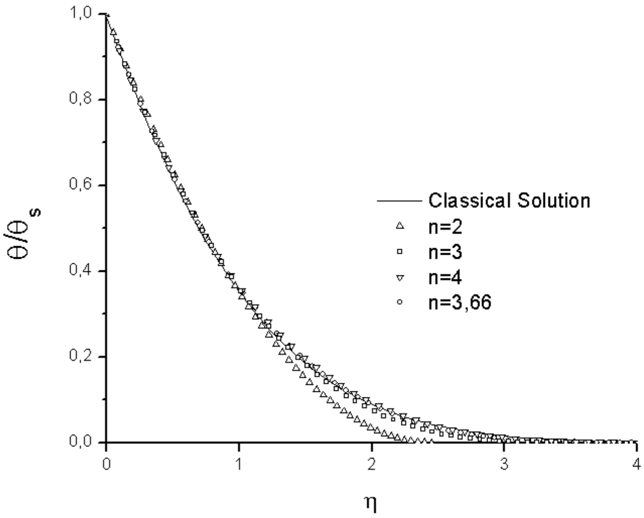

Figure 3. Normalized temperature profiles.

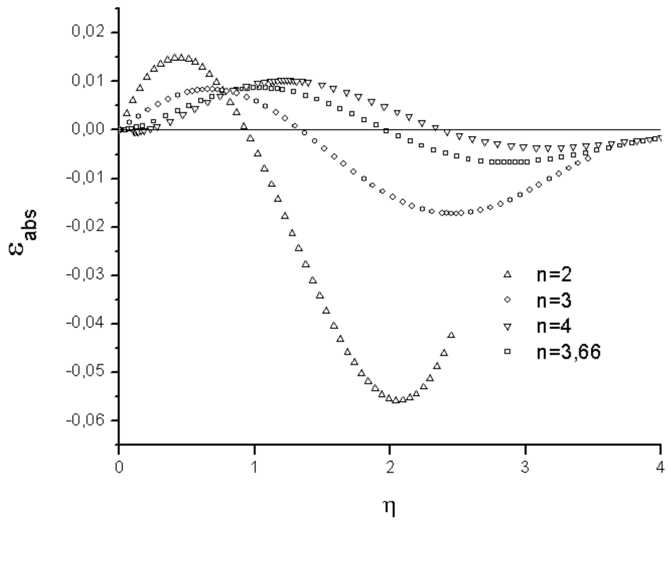

Figure 4. Normalized temperature absolute errors.

Table 3. Maximum (positive and negative) values of the normalized temperature absolute error as function of the *n* value

n value	Positive	Negative
2	0,0149347361	-0,0557545124
3	0,0083527385	-0,0172565845
π / (4- π)	0,0086532853	-0,0066994304
4	0,0102142975	-0,0036899413

For the prescribed heat flux case, the surface temperature can be compared substituting $\eta = 0$ in Eqs. 19 and 21. The following expression is obtained, respectively:

$$\theta_S = \frac{2Q_F\sqrt{\tau}}{\sqrt{\pi}}, \tag{35}$$

$$\theta_S = Q_F\sqrt{\tau}\sqrt{\frac{(n+1)}{n}}. \tag{36}$$

The relative error for the prediction of the surface temperature can be obtained through these expressions ((Eq. 35 – Eq 36)/Eq. 35) resulting in the equation:

$$\varepsilon = 1 - \frac{2}{\sqrt{\pi}}\sqrt{\frac{n}{(n+1)}}, \tag{37}$$

where ε represents the relative error. Figure 5 presents the graphic of this error against the *n* parameter, using Eq. 37. It can be observed in this figure that the error vanishes for the *n* value calculated through the energy balance. It means that the HBIM tracks exactly the surface temperature when this value of *n* is used.

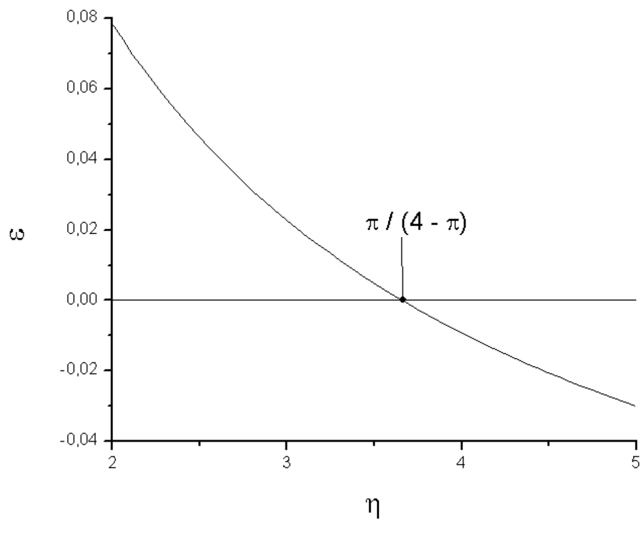

Figure 5. Relative error of the surface temperature against the *n* parameter

B. Prescribed Temperature

The classical prescribed temperature boundary condition solution is given by Eq. 20, while the solution for the HBIM case is given by Eq. 22. The best value of the parameter was also calculated, given by Eq. 34 ($n \approx 1,751938393$).

Figure 6 presents the graphic of Eqs. 20 and 22, both normalized by the prescribed surface temperature, for $n=1$, $n=2$, $n=3$ and Eq. 34 *n* value. In Fig. 7, the absolute relative error for the HBIM solution, defined as *Eq. 20 – Eq. 22* is presented. The maximum positive and negative values of the error are also tabulated in Table 4.

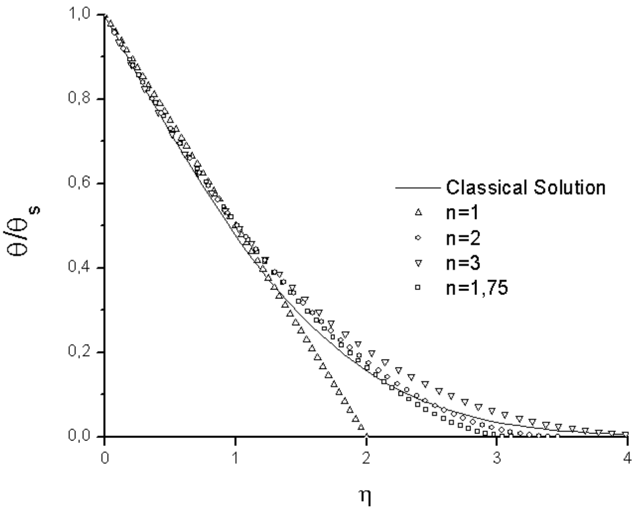

Figure 6. Normalized temperature profiles.

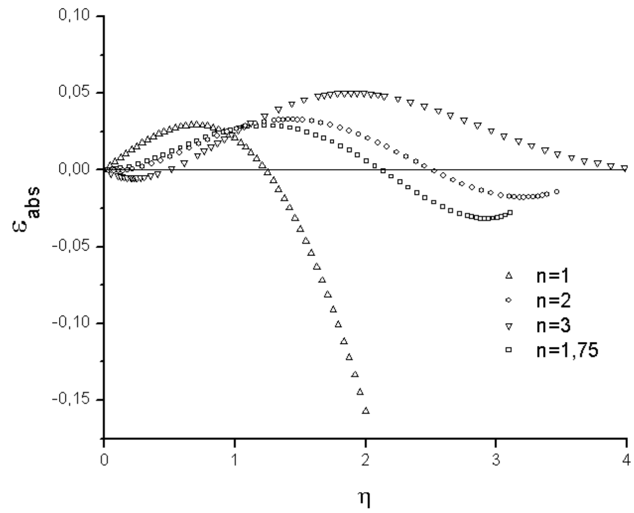

Figure 7. Normalized temperature absolute errors.

Table 4. Normalized temperature maximum absolute error as function of the *n* value

n value	Positive	Negative
1	0,0293841650	-0,1572992070
2 / (π–2)	0,0288509287	-0,03178750671
2	0,0328615882	-0,01783997939
3	0,0502997213	-0,0051436997

In Fig. 6, it can be observed that none of the selected profiles captures the behavior of the classical solution for larges values of η, but among them, the Eq. 34 n value presents the smaller maximum relative errors, as it can be observed in Fig. 7 and in Table 4.

VII. Conclusion

In this paper the Heat Balanced Integral Method was used to solve the heat conduction inside a semi-infinite solid body subjected to two different boundary conditions: prescribed heat flux and prescribed temperature. As part of the method, an n degree function was selected as representative of the temperature distribution of the material.

The value of the n exponent (Eq. 10) was calculated using an energy balance, based on the temperature distribution obtained from the classical solution temperature distribution profile, which was compared with other n values. The comparison shows that the calculated value of n is the best one to be used in these problems.

It is important to note that the calculated value of n is very dependent on the boundary condition. Due to this fact, the authors believe that a time-constant n value is not the best choice to a time-dependent boundary condition problem, but this aspect needs further investigation.

The other cases mentioned in Table 1, which were not studied in the present work needs to be investigated. This is the case of the convection boundary condition, as illustrated in Fig. 1. As already mentioned in the Introduction section, a constant value of the parameter n was adopted, for the previous modeling developed by the authors, which included ablation. The methodology presented in this paper should be used to improve the other results previously published in the literature.

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